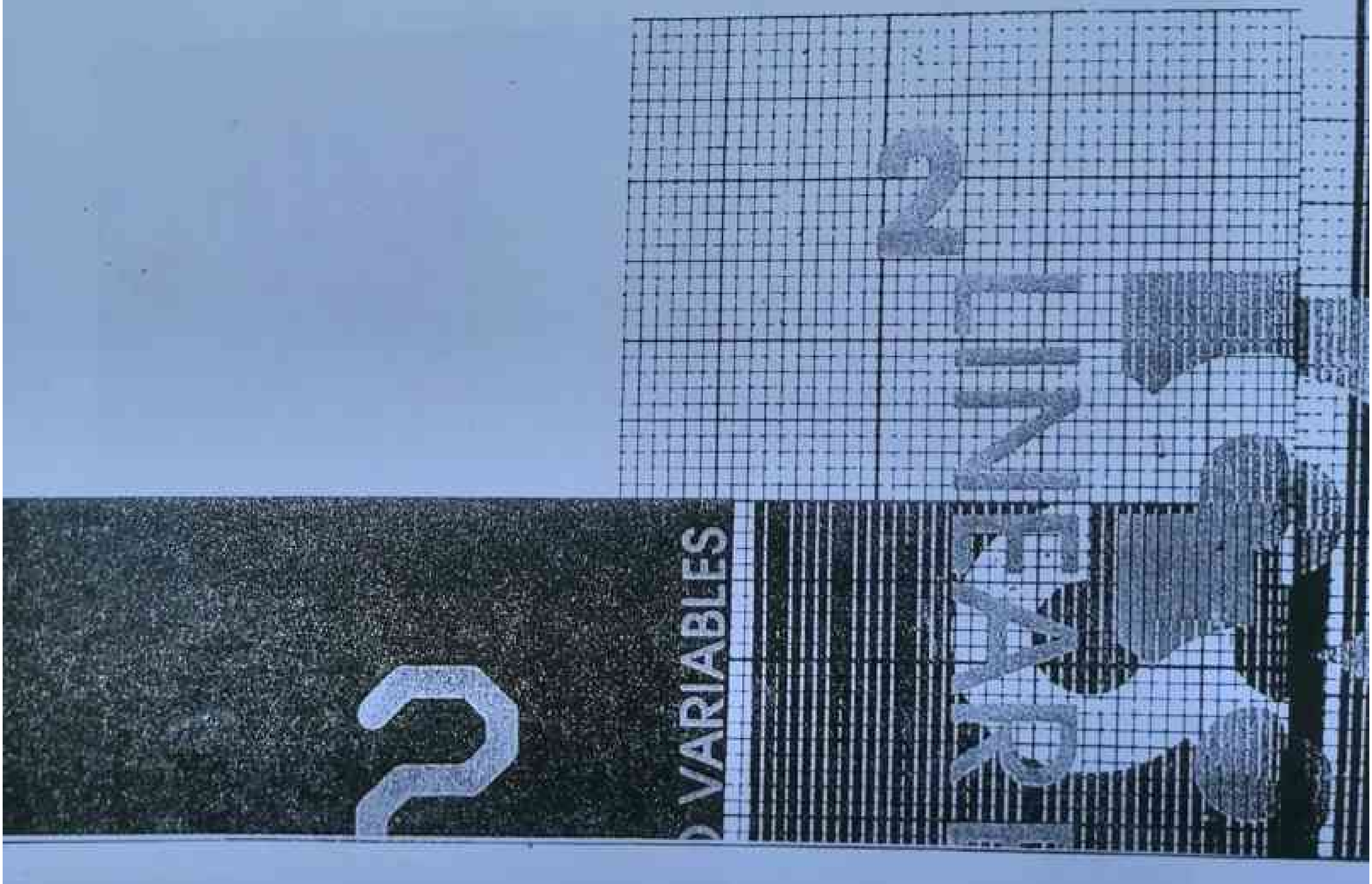
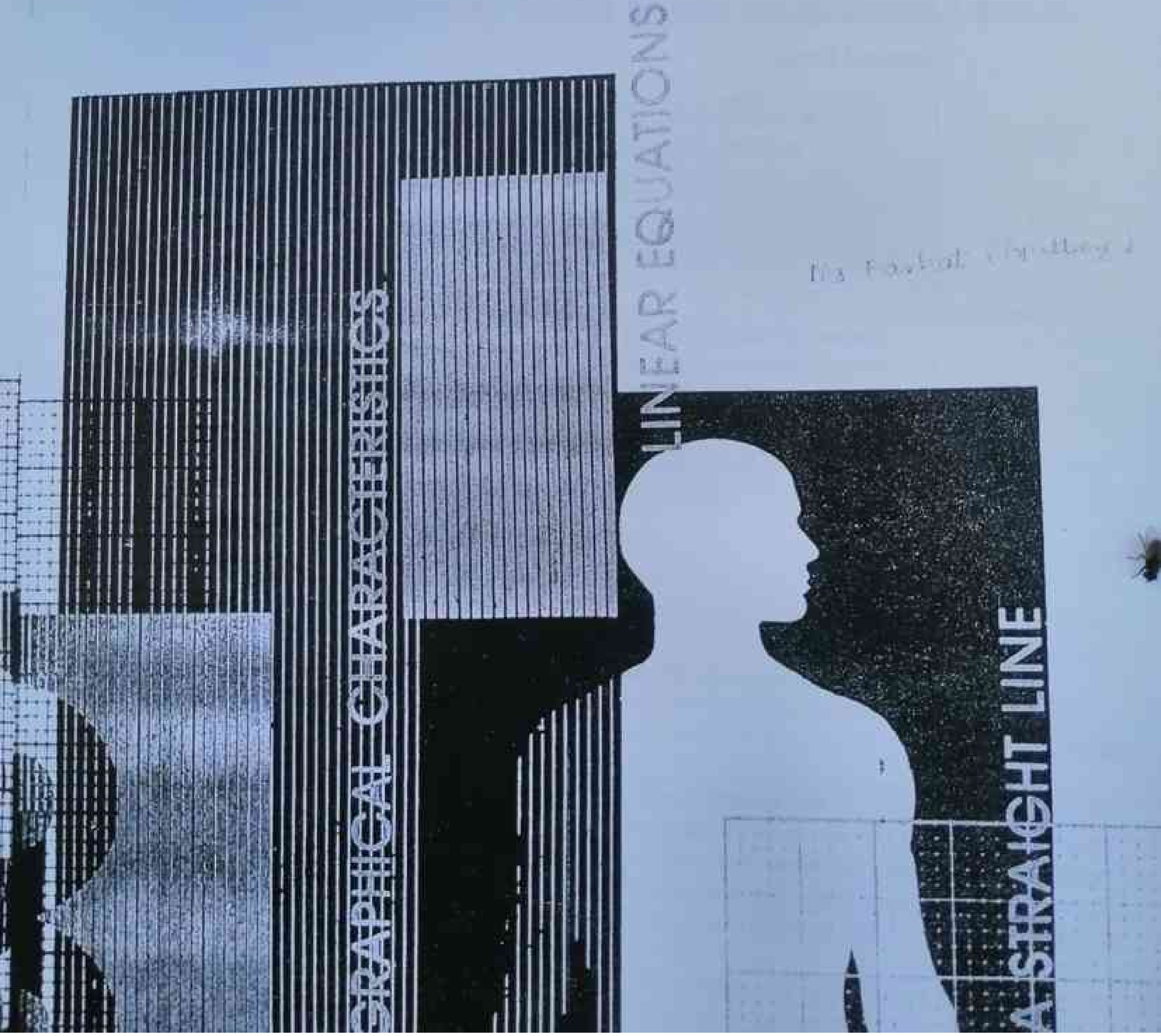
CHAPTER 2

LINEAR EQUATIONS



CHAPTER OBJECTIVES

- ❑ Provide a thorough understanding of the algebraic and graphical characteristics of linear equations
- ❑ Provide the tools which will allow one to determine the equation which represents a linear relationship
- ❑ Illustrate a variety of applications of linear equations



MOTIVATING SCENARIO: Nuclear Power Utilization

Nuclear power accounts for an increasing share of electricity produced in the United States. Annual data gathered by the North American Electricity Council will be presented which estimates the percentage of electricity generated by nuclear power sources during the 1980s. *Using selected data points, the council wants to determine an estimating equation which could be used to forecast future contributions made by nuclear power sources* (Example 19).*

The study of linear mathematics is significant for a number of reasons. First, many real-world phenomena, which we might be interested in representing mathematically, either are linear or can be approximated reasonably well using linear relationships. As a result, linear mathematics is widely applied. Second, the analysis of linear relationships is generally easier than that of nonlinear relationships. Lastly, the methods of analyzing nonlinear relationships are sometimes similar to, or extensions of, those used in linear mathematics. Consequently, a good understanding of linear mathematics is prerequisite to the study of nonlinear mathematics.

2.1 CHARACTERISTICS OF LINEAR EQUATIONS

General Form

LINEAR EQUATION WITH TWO VARIABLES

A linear equation involving two variables x and y has the standard form

$$ax + by = c \tag{2.1}$$

where a, b, and c are constants and a and b cannot *both* equal zero.

Linear equations are *first-degree* equations. Each variable in the equation is raised (implicitly) to the first power. The presence of terms having exponents other than 1 (e.g., x^2) or of terms involving a product of variables (e.g., $2xy$) would exclude an equation from being considered linear.

The following are all examples of linear equations involving two variables:

* Most chapters begin with a **Motivating Scenario.** Their purpose is to provide an example of the type of application a student should be able to solve *upon completion* of the chapter.

	Eq. (2.1) Parameters		
	a	b	c
$2x + 5y = -5$	2	5	−5
$-x + \frac{1}{2}y = 0$	−1	$\frac{1}{2}$	0
$x/3 = 25$	$\frac{1}{3}$	0	25
$\sqrt{2}u - 0.05v = 3.76$	$\sqrt{2}$	−0.05	3.76
$2s - 4t = -\frac{1}{2}$	2	−4	$-\frac{1}{2}$

(*Note:* The names of the variables may be different from x and y.)

The following are examples of equations which are *not* linear. Can you explain why?

$$2x + 3xy - 4y = 10$$
$$x + y^2 = 6$$
$$\sqrt{u} + \sqrt{v} = -10$$
$$ax + \frac{b}{y} = c$$

The form of an equation may not always be obvious. Initially, the equation

$$2x = \frac{5x - 2y}{4} + 10$$

might not appear to be linear. However, multiplying both sides of the equation by 4 and moving all variables to the left side yields $3x + 2y = 40$, which is in the form of Eq. (2.1).

Representation Using Linear Equations

Given a linear equation having the form $ax + by = c$, the ***solution set*** for the equation is the set of all ordered pairs (x, y) which satisfy the equation. Using **set notation** the solution set S can be specified as

$$\boxed{S = \{(x, y) | ax + by = c\}} \tag{2.2}$$

Verbally, this set notation states that the **solution set** S consists of **elements** (x, y) *such that* (the vertical line) the equation $ax + by = c$ is satisfied. Stated differently, Eq. (2.2) expresses that S consists of all **ordered pairs** (x, y) such that $ax + by = c$. For any linear equation, S consists of an infinite number of elements; that is, **there is an infinite number of pairs of values (x, y) which satisfy a linear equation having the form $ax + by = c$.**

To determine any pair of values which satisfy a linear equation, assume a value for one of the variables, substitute this value into the equation, and solve for the corresponding value of the other variable. This method assumes that both variables are included in the equation (i.e., $a \neq 0$ and $b \neq 0$).

EXAMPLE 1 Given the equation

$$2x + 4y = 16$$

(a) Determine the pair of values which satisfies the equation when $x = -2$.
(b) Determine the pair of values which satisfies the equation when $y = 0$.

SOLUTION

(a) Substituting $x = -2$ into the equation, we have

$$2(-2) + 4y = 16$$
$$4y = 20$$
$$y = 5$$

When $x = -2$, the pair of values satisfying the equation is $x = -2$ and $y = 5$, or $(-2, 5)$.
(b) Substituting $y = 0$ into the equation gives

$$2x + 4(0) = 16$$
$$2x = 16$$
$$x = 8$$

When $y = 0$, the pair of values satisfying the equation is $(8, 0)$.

EXAMPLE 2 **(Production Possibilities)** A company manufactures two different products. For the coming week 120 hours of labor are available for manufacturing the two products. Work-hours can be allocated for production of either product. In addition, since both products generate a good profit, management is interested in using all 120 hours during the week. Each unit produced of product A requires 3 hours of labor and each unit of product B requires 2.5 hours.
(a) Define an equation which states that total work-hours used for producing x units of product A and y units of product B equal 120.
(b) How many units of product A can be produced if 30 units of product B are produced?
(c) If management decides to produce one product only, what is the maximum quantity which can be produced of product A? The maximum of product B?

SOLUTION

(a) We can define our variables as follows:

x = number of units produced of product A
y = number of units produced of product B

The desired equation has the following structure.

Total hours used in producing products A and B = 120

(2.3)

More specifically,

Total hours used in producing product A	+	total hours used in producing product B	= 120	(2.4)

Since the total hours used in producing a product equals hours required per unit produced times number of units produced, Eq. (2.4) reduces to

$$\boxed{3x + 2.5y = 120} \tag{2.5}$$

(b) If 30 units of product B are produced, then $y = 30$. Therefore

$$3x + 2.5(30) = 120$$
$$3x = 45$$
$$x = 15$$

Thus, one pair of values satisfying Eq. (2.5) is (15, 30). In other words, *one combination* of the two products which will fully utilize the 120 hours is 15 units of product A and 30 units of product B.

(c) If management decides to manufacture product A only, no units of product B are produced, or $y = 0$. If $y = 0$,

$$3x + 2.5(0) = 120$$
$$3x = 120$$
$$x = 40$$

Therefore 40 is the maximum number of units of product A which can be produced using the 120 hours.

If management decides to manufacture product B only, $x = 0$ and

$$3(0) + 2.5y = 120$$

or

$$y = 48 \text{ units}$$

EXAMPLE 3 We stated earlier that there is an infinite number of pairs of values (x, y) which satisfy any linear equation. In Example 2, are there any members of the solution set which might not be realistic in terms of what the equation represents?

SOLUTION

In Example 2, x and y represent the number of units produced of the two products. Since *negative* production is not possible, negative values of x and y are not meaningful. There are negative values which satisfy Eq. (2.5). For instance, if $y = 60$, then

$$3x + 2.5(60) = 120$$
$$3x = -30$$
$$x = -10$$

In addition to negative values, it is possible to have decimal or fractional values for x and y. For example, if $y = 40$,

$$3x + 2.5(40) = 120$$
$$3x = 20$$
$$x = 6\tfrac{2}{3}$$

Depending upon the nature of the products and how they are sold, fractional values may or may not be acceptable. □

POINT FOR THOUGHT & DISCUSSION Give examples of types of products where only integer values would be reasonable. Give an example of a product for which noninteger values are reasonable.

Linear Equations with n Variables

LINEAR EQUATION WITH n VARIABLES
A linear equation involving n variables $x_1, x_2, x_3, \ldots, x_n$ has the general form

$$a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b \tag{2.6}$$

where $a_1, a_2, a_3, \ldots, a_n$ and b are constants and *not all* $a_1, a_2, a_3, \ldots, a_n$ equal zero.

Each of the following is an example of a linear equation:

$$3x_1 - 2x_2 + 5x_3 = 0$$
$$-x_1 + 3x_2 - 4x_3 + 5x_4 - x_5 + 2x_6 = -80$$
$$5x_1 - x_2 + 4x_3 + x_4 - 3x_5 + x_6 - 3x_7 + 10x_8 - 12x_9 + x_{10} = 1{,}250$$

Given a linear equation involving n variables, as defined by Eq. (2.6), the solution set S can be specified as

$$S = \{(x_1, x_2, x_3, \ldots, x_n) | a_1x_1 + a_2x_2 + a_3x_3 + \cdots + a_nx_n = b\} \tag{2.7}$$

As with the two-variable case, there are infinitely many elements in the solution set. An element in S is represented by a collection of values $(x_1, x_2, x_3, \ldots, x_n)$, one for each of the n variables in the equation. One way of identifying specific elements in S is to assume values for $n-1$ of the variables, substitute these into the equation, and solve for the value of the remaining variable.

EXAMPLE 4 Given the equation

$$2x_1 + 3x_2 - x_3 + x_4 = 16$$

(a) What values satisfy the equation when $x_1 = 2$, $x_2 = -1$, and $x_3 = 0$?
(b) Determine all members of the solution set which have values of 0 for three of the four variables.

SOLUTION
(a) Substituting the given values for x_1, x_2, and x_3 into the equation yields

$$2(2) + 3(-1) - (0) + x_4 = 16$$

or

$$x_4 = 15$$

The corresponding element of the solution set is $(2, -1, 0, 15)$.
(b) If $x_1 = x_2 = x_3 = 0$, then

$$2(0) + 3(0) - (0) + x_4 = 16$$

or

$$x_4 = 16$$

If $x_1 = x_2 = x_4 = 0$,

$$2(0) + 3(0) - x_3 + (0) = 16$$

or

$$x_3 = -16$$

If $x_1 = x_3 = x_4 = 0$, then

$$2(0) + 3x_2 - (0) + (0) = 16$$

or

$$3x_2 = 16$$

and

$$x_2 = \tfrac{16}{3}$$

If $x_2 = x_3 = x_4 = 0$,

$$2x_1 + 3(0) - (0) + (0) = 16$$

or

$$2x_1 = 16$$

and

$$x_1 = 8$$

Therefore, the elements of the solution set which have three of the four variables equaling 0 are $(0, 0, 0, 16)$, $(0, 0, -16, 0)$, $(0, \tfrac{16}{3}, 0, 0)$, and $(8, 0, 0, 0)$. ☐

elements in S is to assume values for $n-1$ of the variables, substitute these into the equation, and solve for the value of the remaining variable.

EXAMPLE 4 Given the equation

$$2x_1 + 3x_2 - x_3 + x_4 = 16$$

(*a*) What values satisfy the equation when $x_1 = 2$, $x_2 = -1$, and $x_3 = 0$?
(*b*) Determine all members of the solution set which have values of 0 for three of the four variables.

SOLUTION

(*a*) Substituting the given values for x_1, x_2, and x_3 into the equation yields

$$2(2) + 3(-1) - (0) + x_4 = 16$$

or

$$x_4 = 15$$

The corresponding element of the solution set is $(2, -1, 0, 15)$.
(*b*) If $x_1 = x_2 = x_3 = 0$, then

$$2(0) + 3(0) - (0) + x_4 = 16$$

or

$$x_4 = 16$$

If $x_1 = x_2 = x_4 = 0$,

$$2(0) + 3(0) - x_3 + (0) = 16$$

or

$$x_3 = -16$$

If $x_1 = x_3 = x_4 = 0$, then

$$2(0) + 3x_2 - (0) + (0) = 16$$

or

$$3x_2 = 16$$

and

$$x_2 = \tfrac{16}{3}$$

If $x_2 = x_3 = x_4 = 0$,

$$2x_1 + 3(0) - (0) + (0) = 16$$

or

$$2x_1 = 16$$

and

$$x_1 = 8$$

Therefore, the elements of the solution set which have three of the four variables equaling 0 are $(0, 0, 0, 16)$, $(0, 0, -16, 0)$, $(0, \tfrac{16}{3}, 0, 0)$, and $(8, 0, 0, 0)$. ☐

PRACTICE EXERCISE
In Example 2 (Production Possibilities), assume that a third product (product C) is also to be produced. Because of the additional product, management has authorized an additional 30 labor hours. If each unit of product C requires 3.75 labor hours, (*a*) determine the equation which requires that all 150 labor hours be used in producing the three products and (*b*) determine the maximum number of units which could be produced of each product. *Answer:* (*a*) If z = number of units produced of product C, $3x + 2.5y + 3.75z = 150$, (*b*) 50 units of A, 60 units of B, and 40 units of C.

Section 2.1 Follow-up Exercises

Determine which of the following equations are linear.

1 $-3y = 0$

2 $\sqrt{2}x + 6y = -25$

3 $-5x + 24y = 200$

4 $-x^2 + 3y = 40$

5 $2x - 3xy + 5y = 10$

6 $\sqrt{4x} - 3y = -45$

7 $u - 3v = 20$

8 $r/2 + s/5 = \frac{1}{4}$

9 $m/2 + (2m - 3n)/5 = 0$

10 $(x + 2y)/3 - 3x/4 = 2x - 5y$

11 $40 - 3y = \sqrt{24}$

12 $0.0003x - 2.3245y = x + y - 3.2543$

13 $2x_1 - 3x_2 + x_3 = 0$

14 $(x_1 - 3x_2 + 5x_3 - 2x_4 + x_5)/25 = 300$

15 $(x_1 + x_2 - x_3x_1) = 5$

16 $3x_2 - 4x_1 = 5x_3 + 2x_2 - x_4 + 36$

17 $\sqrt{x^2 + 2xy + y^2} = 25$

18 $(2x_1 - 3x_2 + x_3)/4 = (x_2 - 2x_4)/5 + 90$

19 Consider the equation $8x = 120$ as a two-variable equation having the form of Eq. (2.1).
(*a*) Define *a*, *b*, and *c*.
(*b*) What pair of values satisfies the equation wnen $y = 10$?
(*c*) What pair of values satisfies the equation when $x = 20$?
(*d*) Verbalize the somewhat unique nature of the solution set for this equation.

20 Rework Example 2 if product A requires 2 hours per unit and product B requires 4 hours per unit.

21 Given the equation $4x_1 - 2x_2 + 6x_3 = 0$:
(*a*) What values satisfy the equation when $x_1 = 2$ and $x_3 = 1$?
(*b*) Define all elements of the solution set in which the values of two variables equal 0.

22 Given the equation $x_1 - 3x_2 + 4x_3 - 2x_4 = -60$:
(*a*) What values satisfy the equation when $x_1 = 20$, $x_2 = 6$, and $x_3 = -4$?
(*b*) Determine all elements of the solution set for which the values of three variables equal zero.

23 The equation $x_3 = 20$ is one of a set of related equations involving four variables x_1, x_2, x_3, and x_4.
(*a*) What values satisfy the equation when $x_1 = 4$, $x_2 = 2$, and $x_4 = 15$?
(*b*) What values satisfy the equation when $x_3 = 10$?
(*c*) Determine all elements of the solution set for which the values of three variables equal zero.

24 **Product Mix** A company manufactures two products, A and B. Each unit of A requires 3 labor hours and each unit of B requires 5 labor hours. Daily manufacturing capacity is 150 labor hours.
(*a*) If *x* units of product A and *y* units of product B are manufactured each day and all labor hours are to be used, determine the linear equation that requires the use of 150 labor hours per day.

(b) How many units of A can be made each day if 25 units of B are manufactured each day?

(c) How many units of A can be made each *week* if 12 units of B are manufactured each day? (Assume a 5-day work week.)

25 **Nutrition Planning** A dietitian is considering three food types to be served at a meal. She is particularly concerned with the amount of one vitamin provided at this meal. One ounce of food 1 provides 8 milligrams of the vitamin; an ounce of food 2 provides 24 milligrams; and, an ounce of food 3 provides 16 milligrams. The *minimum daily requirement* (MDR) for the vitamin is 120 milligrams.

(a) If x_j equals the number of ounces of food type j served at the meal, determine the equation which ensures that the meal satisfies the MDR exactly.

(b) If only one of the three food types is to be included in the meal, how much would have to be served (in each of the three possible cases) to satisfy the MDR?

26 **Emergency Airlift** The Red Cross wants to airlift supplies into a South American country which has experienced an earthquake. Four types of supplies, each of which would be shipped in containers, are being considered. One container of a particular item weighs 120, 300, 250, and 500 pounds, respectively, for the four items. If the airplane to be used has a weight capacity of 60,000 pounds and x_j equals the number of containers shipped of item j:

(a) Determine the equation which ensures that the plane will be loaded to its weight capacity.

(b) If it is decided to devote this plane to one supply item only, how many containers could be shipped of each item?

27 **Airlift Revisited** In Exercise 26, each container of a supply item requires a specific volume of space. Suppose containers of the four items require 30, 60, 50, and 80 cubic feet, respectively. If the volume capacity of the plane is 15,000 cubic feet:

(a) Determine the equation which ensures that the volume capacity of the plane is filled exactly.

(b) If it is decided to devote this plane to one supply item only, how many containers could be shipped of each item if volume capacity is the only consideration?

(c) Using the information from Exercise 26, what is the maximum number of containers which could be shipped of each item if both weight and volume are considered? Indicate in each case whether weight or volume capacity is the constraining factor.

28 **Personnel Hiring** A software consulting firm has received a large contract to develop a new airline reservation system for a major airline. In order to fulfill the contract, new hiring of programmer analysts, senior programmer analysts, and software engineers will be required. Each programmer analyst position will cost $40,000 in salary and benefits. Each senior programmer analyst position will cost $50,000 and each software engineer position $60,000. The airline has budgeted $1.2 million per year for the new hirings. If x_j equals the number of persons hired for job category j (where $j = 1$ corresponds to programmer analysts, etc.):

(a) Determine the equation which ensures that total new hires will exactly consume the budget.

(b) If it were desired to spend the entire budget on one type of position, how many persons of each type could be hired?

(c) If exactly 10 programmer analysts are needed for the contract, what is the maximum number of senior programmer analysts that could be hired? Maximum number of software engineers?

29 **Public Transportation** New York City has received a federal grant of $100 million for improving public transportation. The funds are to be used only for the purchase of new buses, the purchase of new subway cars, or the repaving of city streets. Costs are

estimated at $150,000 per bus, $180,000 per subway car, and $250,000 per mile for repaving. City officials want to determine different ways of spending the grant money.
(a) Define the decision variables and write the equation which ensures complete expenditure of the federal grant.
(b) If it has been determined that 100 buses and 200 new subway cars will be purchased, how many miles of city streets can be repaved?
(c) If officials wish to spend all of the grant money on one type of improvement, what are the different possibilities?

30 **Political Campaign** A candidate for the position of governor of a midwestern state has an advertising budget of $1.5 million. The candidate's advisors have identified four advertising options: newspaper advertisements, radio commercials, television commercials, and billboard advertisements. The costs for these media options average $1,500, $2,500, $10,000, and $7,500, respectively. If x_j equals the number of units purchased of media option j:
(a) Write an equation which requires total advertising expenditures of $1.5 million.
(b) If it has been determined that 100 newspaper ads, 300 radio ads, and 50 billboard ads will be used, how many television ads can they purchase?
(c) If 50 billboard ads are to be purchased, what is the maximum number of newspaper ads that can be purchased? Maximum number of radio ads? TV ads?

2.2 GRAPHICAL CHARACTERISTICS

Graphing Two-Variable Equations

A linear equation involving two variables graphs as a straight line in two dimensions. To graph this type of linear equation, *(1) identify and plot the coordinates of any two points which lie on the line, (2) connect the two points with a straight line, and (3) extend the straight line in both directions as far as necessary or desirable for your purposes.* The coordinates of the two points are found by identifying any two members of the solution set. Each element in the solution set graphs as a point (x, y) in 2-space, where x and y are the respective values of the two variables. For example, if the values of $x = 1$ and $y = 3$ satisfy an equation, the graphical representation of this member of the solution set is the point (1, 3).

EXAMPLE 5 The graph of the equation

$$2x + 4y = 16$$

is found by first identifying any two pairs of values for x and y which satisfy the equation.

NOTE Aside from the case where the right side of the equation equals 0, the easiest points to identify (algebraically) are those found by setting one variable equal to 0 and solving for the value of the other variable. That is, let $x = 0$ and solve for the value of y; then let $y = 0$ and solve for the value of x. Observe that the resulting ordered pairs, $(0, y)$ and $(x, 0)$, are points on the y and x axes, respectively.

Letting $x = 0$, the corresponding value for y is 4, and letting $y = 0$ results in $x = 8$. Thus (0, 4) and (8, 0) are two members of the solution set, and their graphical representation is

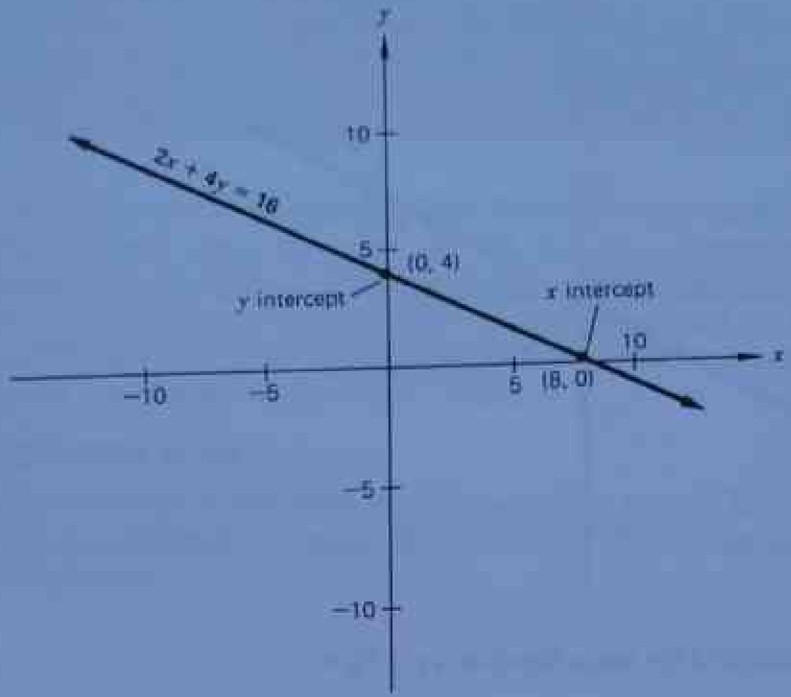


Figure 2.1 Graph of the linear equation $2x + 4y = 16$.

indicated by the two points in Fig. 2.1. The two points have been connected by a straight line, and the line has been extended in both directions.

Just as (0, 4) and (8, 0) are members of the solution set for the equation $2x + 4y = 16$, the coordinates of every point lying on the line represent other members of the solution set. How many unique points are there on the line? There are infinitely many, which is entirely consistent with our earlier statement that there are an infinite number of pairs of values for x and y which satisfy any linear equation. In summary, all pairs of values (x, y) that belong to the solution set of a linear equation are represented graphically by the points lying on the line representing the equation. In Fig. 2.1, the coordinates of any point *not* lying on the line are not members of the solution set for $2x + 4y = 16$.

EXAMPLE 6 Graph the linear equation $4x - 7y = 0$.

SOLUTION

This equation is an example where two different points will not be identified by setting each variable equal to 0 and solving for the remaining variable. Watch what happens! If $x = 0$,

$$4(0) - 7y = 0 \qquad \text{or} \qquad y = 0$$

If $y = 0$,

$$4x - 7(0) = 0 \qquad \text{or} \qquad x = 0$$

Both cases have yielded the same point, (0, 0). Therefore, to identify a second point, a value other than zero must be assumed for one of the variables. If we let $x = 7$,

$$4(7) - 7y = 0$$
$$-7y = -28$$
$$y = 4$$

Two members of the solution set, then, are (0, 0) and (7, 4). Figure 2.2 illustrates the graph of the equation. ☐

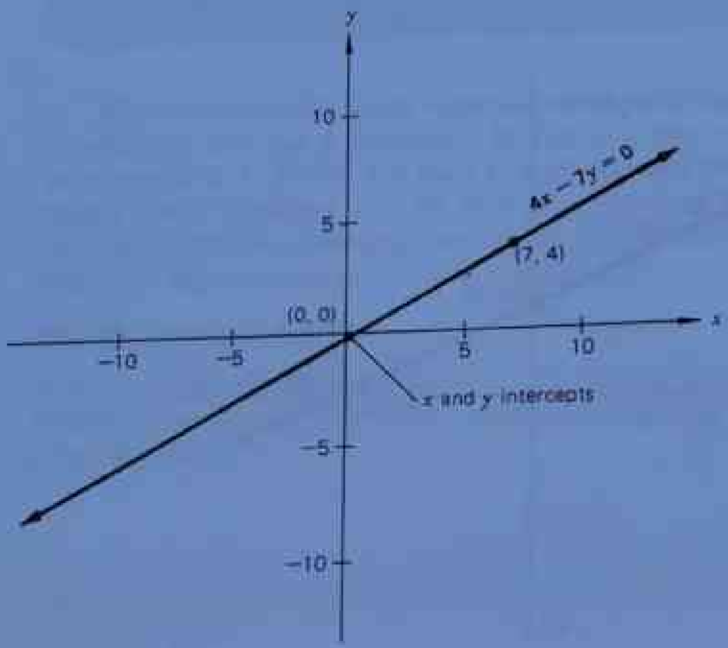


Figure 2.2 Graph of the linear equation $4x - 7y = 0$.

> **NOTE** Any two-variable linear equation having the form $ax + by = 0$ graphs as a straight line which *passes through the origin.* The unique property of this equation is that the right side, c, equals zero.

Intercepts

In describing the graphical appearance of a straight line, two significant attributes are the ***x* intercept** and ***y* intercept.** These can be described both graphically and algebraically.

> **DEFINITION: *x* INTERCEPT**
> The x intercept of an equation is the point where the graph of the equation crosses the x axis. The x intercept represents the ordered pairs found by setting $y = 0$.

DEFINITION: y INTERCEPT
The y intercept of an equation is the point where the graph of the equation crosses the y axis. The y intercept represents the ordered pairs found by setting $x = 0$.

For a two-variable linear equation there exist (except for two special cases) one x intercept and one y intercept. In Fig. 2.1, the x intercept is (8, 0), and the y intercept is (0, 4). In Fig. 2.2, the x and y intercepts both occur at the same point, the origin. The x intercept is (0, 0), and the y intercept is (0, 0). Examine both figures and verify that the x intercept represents a point having a y value of 0 and that the y intercept represents a point with x value of 0.

The Equation $x = k$

A linear equation of the form $ax = c$ is a special case of Eq. (2.1) where $b = 0$. For this equation there is no y term. Dividing both sides of the equation by a yields the simplified form

$$x = c/a$$

Since c and a are constants, we can let $c/a = k$ and write the equation in the equivalent form

$$\boxed{x = k} \qquad (2.8)$$

where k is a real number. This linear equation is special in the sense that x equals k regardless of the value of y. Perhaps this is understood more easily if Eq. (2.8) is rewritten as

$$x + 0y = k$$

The variable y may assume any value as long as $x = k$. That is the only condition required by the equation. As a result, *any equation of this form graphs as a vertical line crossing the x axis at $x = k$.* Figure 2.3 illustrates two equations of this type. *Note that for these equations there is an x intercept* $(k, 0)$ *but no y intercept* (unless $k = 0$). What happens when $k = 0$?

The Equation $y = k$

A linear equation of the form $by = c$ is also a special case of Eq. (2.1) where $a = 0$; i.e., there is no x term. After both sides of the equation are divided by b, the general reduced form of this case is

$$\boxed{y = k} \qquad (2.9)$$

where k is a real number. This equation indicates that y equals k for any value of x. Again, we can see this more readily if Eq. (2.9) is rewritten as

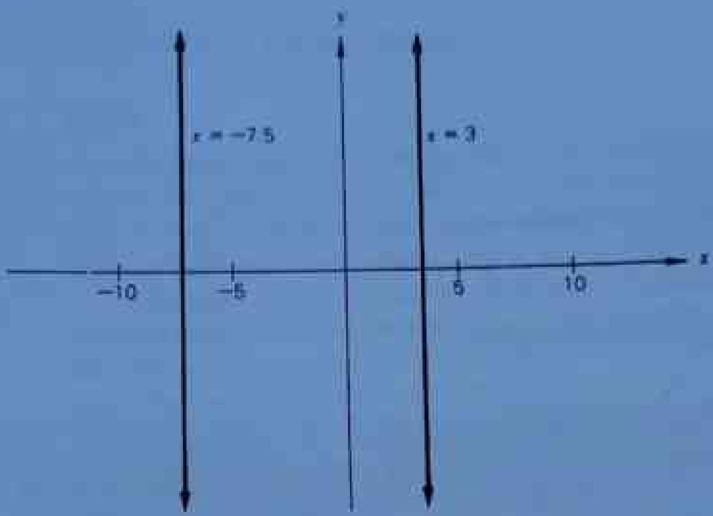


Figure 2.3 Graphs of sample equations of the form $x = k$.

$$0x + y = k$$

The variable x may assume any value as long as $y = k$. *Any equation of this form graphs as a horizontal line crossing the y axis at $y = k$.* Figure 2.4 illustrates two such equations. *Note that equations of this form have no x intercepts* (unless $k = 0$). What happens when $k = 0$?

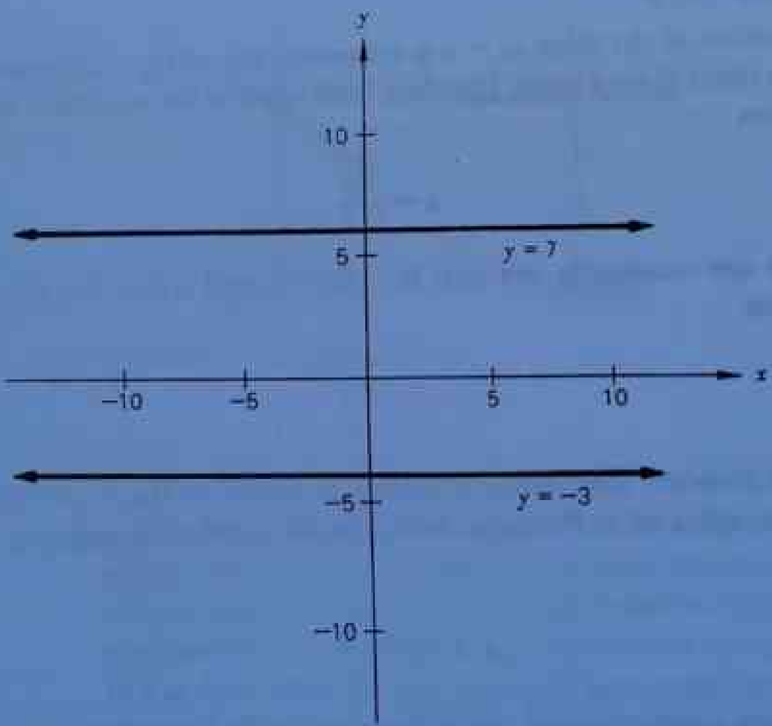


Figure 2.4 Graphs of sample equations of the form $y = k$.

Slope

Any straight line, with the exception of vertical lines, can be characterized by its **slope.** By "slope" we are referring to the *inclination of a line*—whether it rises or falls as you move from left to right along the x axis—and *the rate at which the line rises or falls* (in other words, how steep the line is).

The slope of a line may be ***positive, negative, zero,*** *or* ***undefined.*** **A line with a positive slope rises from left to right, or runs uphill.** For such a line the value of y increases as x increases (or conversely, y decreases as x decreases). **A line having a negative slope falls from left to right, or runs downhill.** For such a line the value of y decreases as x increases (or conversely y increases as x decreases). This means that x and y behave in an **inverse** manner; as one increases, the other decreases and vice versa. **A line having a zero slope is horizontal.** As x increases or decreases, y stays constant (the special case: $y = k$). **Vertical lines (of the form $x = k$) have a slope which is undefined.** Since x is constant, we cannot observe the behavior of y as x changes. These slope relationships are illustrated in Fig. 2.5.

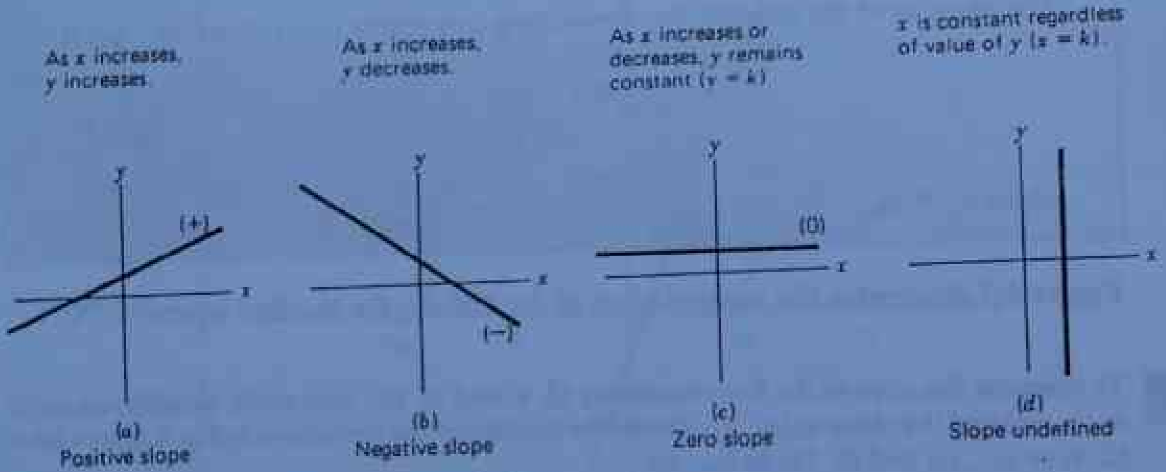


Figure 2.5 Slope conditions for straight lines.

The slope of a line is quantified by a real number. The sign of the slope (number) indicates whether the line is rising or falling. The magnitude (absolute value) of the slope indicates the relative steepness of the line. *The slope tells us the rate at which the value of y changes* relative to *changes in the value of x.* The larger the absolute value of the slope, the steeper the angle at which the line rises or falls. In Fig. 2.6*a* lines *AB* and *CD* both have positive slopes, but the slope of *CD* is larger than that for *AB*. Similarly, in Fig. 2.6*b* lines *MN* and *OP* both have negative slopes, but *OP* has the larger slope in an absolute value sense; it is more steeply sloped.

Given any two points which lie on a (nonvertical) straight line, the slope can be computed as a ratio of the change in the value of y while moving from one point to the other divided by the corresponding change in the value of x, or

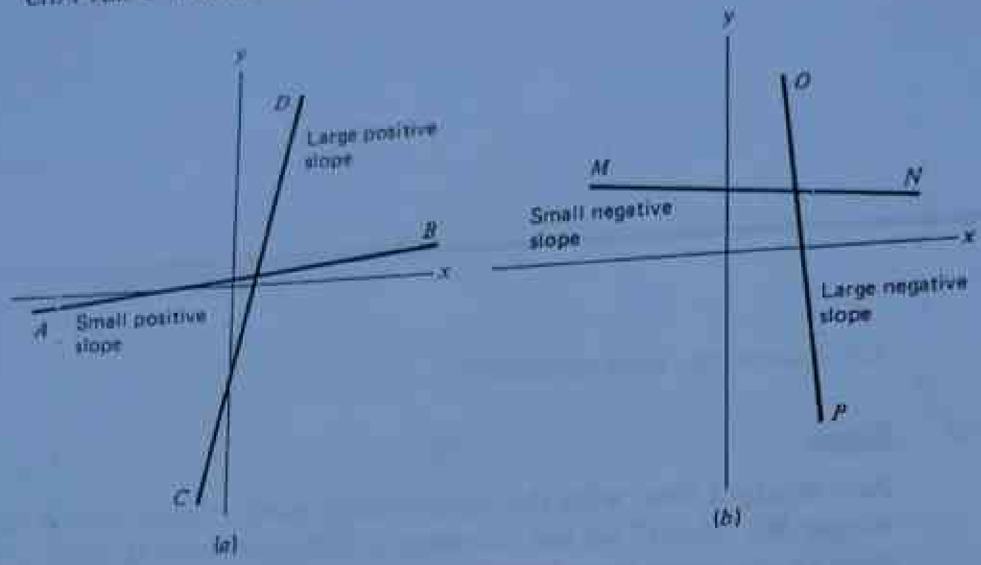


Figure 2.6 Slope: Comparing relative steepness.

$$\text{Slope} = \frac{\text{change in } y}{\text{change in } x} = \frac{\Delta y}{\Delta x}$$

where Δ (delta) means "change in." Thus Δy denotes "the change in the value of y" and Δx "the change in the value of x." The ***two-point formula*** is one way of determining the slope of a straight line connecting two points.

TWO-POINT FORMULA

The slope m of the straight line connecting two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{2.10}$$

where $x_1 \neq x_2$.

Figure 2.7 illustrates the computation of Δx and Δy for the line segment PQ.

EXAMPLE 7 To compute the slope of the line connecting (2, 4) and (5, 12), arbitrarily identify one point as (x_1, y_1) and the other as (x_2, y_2). Given the location of the two points in Fig. 2.8, let's label (2, 4) as (x_1, y_1) and (5, 12) as (x_2, y_2).

In moving from (2, 4) to (5, 12), the change in the value of y is

$$\Delta y = y_2 - y_1 = 12 - 4 = 8$$

Figure 2.7
Measuring Δx and Δy.

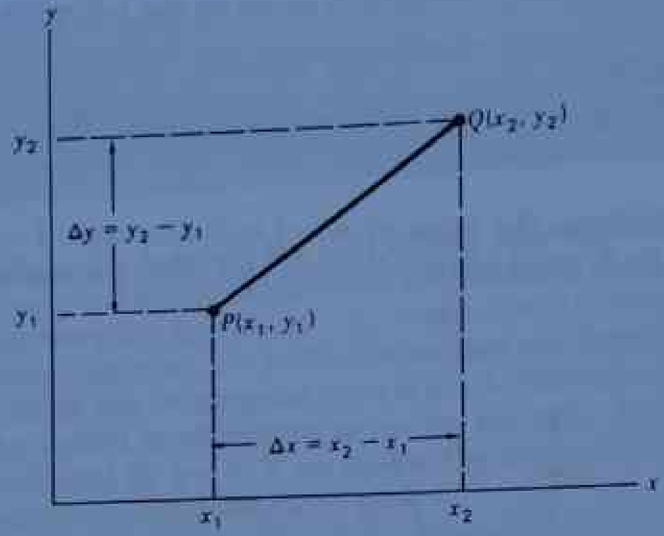
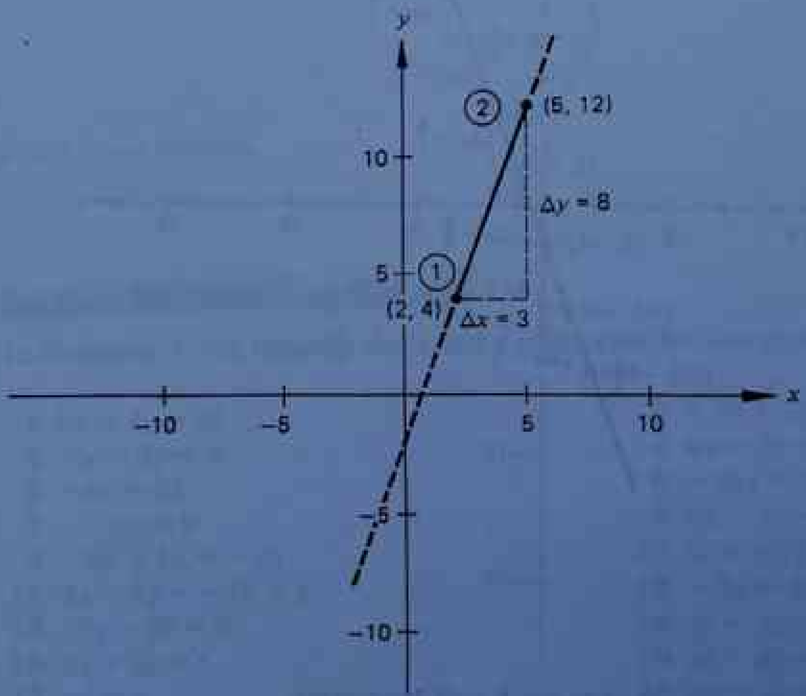


Figure 2.8



Similarly, the corresponding change in x is

$$\Delta x = x_2 - x_1 = 5 - 2 = 3$$

Hence,

$$m = \frac{\Delta y}{\Delta x} = \frac{8}{3}$$

The slope is positive, indicating that the line segment rises from left to right. The sign combined with the magnitude indicate that in moving along the line segment, y increases at a rate of 8 units for every 3 units that x increases.

❑

PRACTICE EXERCISE

Verify that the result from Eq. (2.10) is unaffected by the choice of (x_1, y_1) and (x_2, y_2). In Example 7, label (5, 12) as (x_1, y_1) and (2, 4) as (x_2, y_2) and recompute the slope.

Another way of interpreting the slope is given by the following definition.

DEFINITION: SLOPE

The *slope* is the change in the value of y if x increases by 1 unit.

According to this definition, the value of $m = \frac{8}{3}$ indicates that if x increases by 1 unit, y will *increase* by $\frac{8}{3}$ or $2\frac{2}{3}$ units. Observe this with the sequence of points identified in Fig. 2.9.

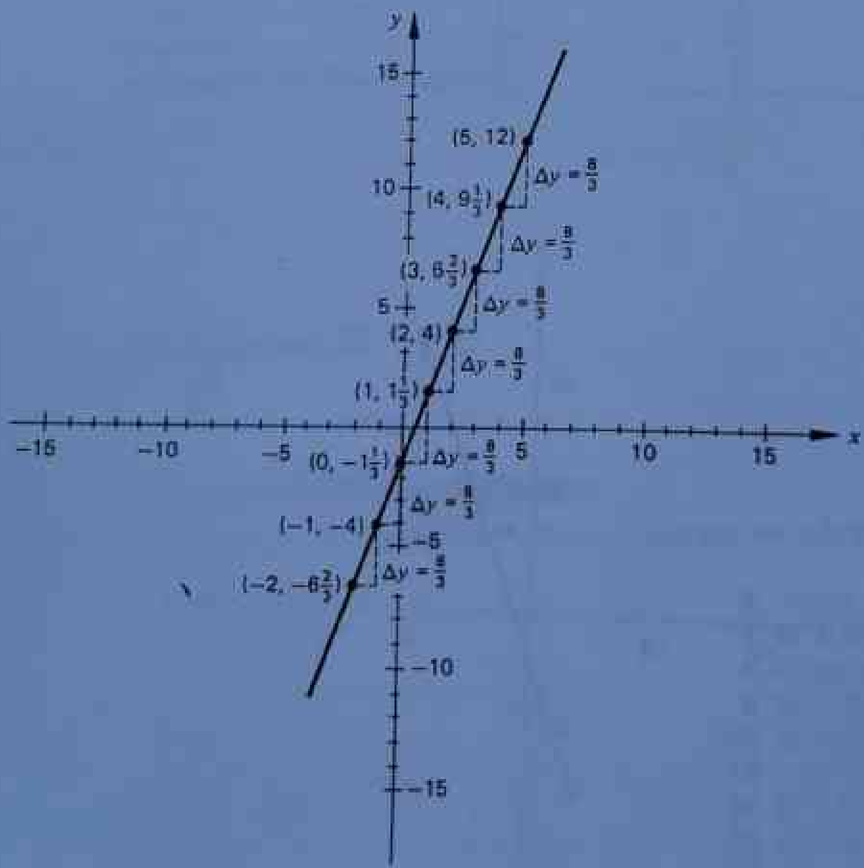


Figure 2.9 y increases by $\frac{8}{3}$ for each unit x increases.

NOTE Along any straight line the slope is constant. That is, if a line is said to have a slope of −2, the slope of the line segment connecting any two points on the line will always equal −2.

EXAMPLE 8 **(Slope Undefined)** We have already seen that the slope of a linear equation having the form $y = k$ is 0. For a horizontal line, the value of y is always the same, and the numerator of the two-point formula, $y_2 - y_1$, always equals 0. We also examined the other special case of a linear equation, $x = k$. We verified that any linear equation having this form graphs as a vertical line crossing the x axis at $x = k$. *The slope of any vertical line is undefined.* This can be verified by attempting to use the two-point formula to determine the slope of the line described by $x = 5$. If we choose the two points $(x_1, y_1) = (5, 0)$ and $(x_2, y_2) = (5, -1)$, substitution into Eq. (2.10) gives

$$m = \frac{-1-0}{5-5}$$

$$= \frac{-1}{0}$$

which is not defined. ❑

Section 2.2 Follow-up Exercises

In Exercises 1–20, identify the x and y intercepts for the given linear equation.

1 $3x - 4y = 24$
2 $-2x + 5y = -20$
3 $-x + 3y = 9$
4 $4x + 2y = 36$
5 $-4x = 12$
6 $-10x + 300 = 0$
7 $x - 2y = 0$
8 $5x - 3y = 0$
9 $-8x + 5y = -20$
10 $(x + y)/2 = 3x - 2y + 16$
11 $2x - 3y = -18 + x$
12 $-3x + 4y - 10 = 7x - 2y + 50$
13 $15y - 90 = 0$
14 $(x - 2y)/3 - 12 = (2x + 4y)/3$
15 $ax + by = t$
16 $cx - dy = e$
17 $px = q$
18 $dx - ey + f = gx - hy$
19 $-ry = s$
20 $-e + fx - gy = h$

For Exercises 21–36, graph the given linear equation.

21 $2x - 3y = -12$
22 $-3x + 6y = -30$
23 $x - 2y = -8$
24 $-8x + 3y = 24$
25 $-x - 4y = 10$
26 $4x + 3y = -36$
27 $3x + 8y = 0$
28 $10x - 5y = 0$
29 $-5x + 2y = 0$
30 $8x - 4y = 0$
31 $-4x = 24$
32 $-2y = -9$
33 $-5y = -17.5$
34 $8x = 20$
35 $-nx = t, n > 0, t > 0$
36 $my = q, m > 0, q < 0$
37 What is the equation of the x axis? The y axis?

In Exercises 38–59, compute the slope of the line segment connecting the two points. Interpret the meaning of the slope in each case.

38 (2, 8) and (−2, −8)
40 (3, 5) and (−1, 15)
42 (−2, 3) and (1, −9)
44 (4, −3) and (10, −12)
46 (−2, 8) and (3, 8)
48 (−4, 20) and (−4, 30)
50 (0, 30) and (0, −15)
52 (a, b) and $(-a, b)$
54 $(d, -c)$ and (0, 0)
56 $(3, b)$ and $(-10, b)$
58 $(a + b, c)$ and (a, c)

39 (−3, 10) and (2, −5)
41 (10, −3) and (12, 4)
43 (5, 8) and (−3, 28)
45 (8, −24) and (5, −15)
47 (−5, −4) and (−5, 6)
49 (5, 0) and (−25, 0)
51 (5, 0) and (0, −10)
53 (0, 0) and (a, b)
55 (−5, −5) and (5, 5)
57 $(-a, -b)$ and $(a, -b)$
59 $(c + d, -c - d)$ and $(a + b, -a - b)$

2.3 SLOPE-INTERCEPT FORM

From a Different Vantage Point

In this section we discuss another form of expressing linear equations. In Sec. 2.1 we stated the general form of a two-variable linear equation as

$$ax + by = c \tag{2.1}$$

Solving Eq. (2.1) for the variable y, we get

$$by = c - ax$$

or

$$\boxed{y = \frac{c}{b} - \frac{ax}{b}} \tag{2.11}$$

For any linear equation the terms c/b and $-a/b$ on the right side of Eq. (2.11) have special significance, provided that $b \neq 0$. If $x = 0$, $y = c/b$. Thus, the term c/b in Eq. (2.11) is the ordinate of the y intercept. Similarly, what happens to the value of y if x increases by one unit in Eq. (2.11)? The value of y changes by $-a/b$. Thus, $-a/b$ is the slope for the equation.

This information is obtained from any linear equation of the form of Eq. (2.1) *if it can be solved for y*. Equation (2.11) is called the ***slope-intercept form*** of a linear equation. Equation (2.11) can be generalized in a simpler form:

$$\boxed{y = mx + k} \tag{2.12}$$

where *m represents the slope* of the *line* and *k* is *the y coordinate of the y intercept.*

To illustrate this form, the equation

$$5x + y = 10$$

can be rewritten in the slope-intercept form as

$$y = -5x + 10$$

Hence, the slope is -5 and the y intercept equals (0, 10).

NOTE Why did the author use the letter k instead of b in Eq. (2.12)? So as not to confuse it with the b in Eq. (2.1)! Students have often seen the slope-intercept form stated as $y = mx + b$.

PRACTICE EXERCISE
Choose two points which satisfy the equation $5x + y = 10$ and verify that the slope equals -5 using Eq. (2.10).

EXAMPLE 9 The equation $y = 2x/3$ can be rewritten in slope-intercept form as

$$y = (\tfrac{2}{3})x + 0$$

The absence of the isolated constant on the right side implicitly suggests that $k = 0$. The graph of this equation is a line having a slope of $\frac{2}{3}$ and a y intercept (0, 0).

EXAMPLE 10 The special case of a linear equation $y = k$ is in the slope-intercept form. To realize this, you must recognize that this equation can be written in the form $y = 0x + k$. The absence of the x term on the right side implicitly suggests that $m = 0$; i.e., the slope of the line having this form equals zero. We confirmed this in Sec. 2.2 when we discussed the graphical characteristics of this case. The y intercept is $(0, k)$ for such equations.

EXAMPLE 11 For the special case $x = k$, it is impossible to solve for the slope-intercept form of the linear equation. The variable y is not a part of the equation. Our conclusion is that it is impossible to determine the slope and y intercept for equations having this form. Look back at Fig. 2.3 to see whether this conclusion is consistent with our earlier findings.

❑

Interpreting the Slope and y Intercept

In many applications of linear equations, the slope and y intercept have meaningful interpretations. Take, for example, the salary equation

$$y = 3x + 25$$

where y = *weekly salary, dollars*
and x = *number of units sold during 1 week*
The salary equation is linear and is written in slope-intercept form. Graphically, the equation is represented by the line in Fig. 2.10, which has a slope of $+3$ and y

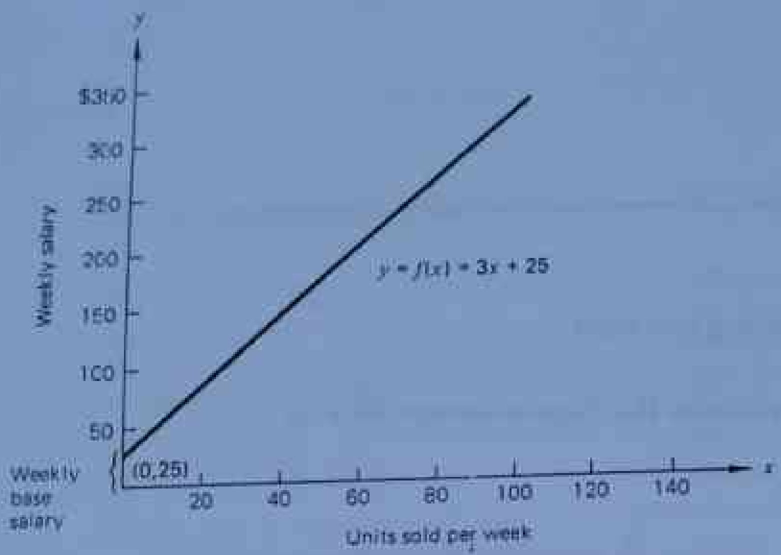


Figure 2.10 Salary function.

intercept equal to (0, 25). Notice that this equation has been graphed only for nonnegative values of x and y. Can you suggest why this would be appropriate?

Think back to the definition of *slope*. Since the slope represents the change in y associated with a unit increase in x, the slope of $+3$ means that weekly salary y increases by $3 for each additional unit sold. The y coordinate of the y intercept represents the value of y when $x = 0$, or the salary which would be earned if no units were sold. This is the base salary per week.

EXAMPLE 12 A police department estimates that the total cost C of owning and operating a patrol car can be estimated by the linear equation

$$C = 0.40x + 18{,}000$$

where C = *total cost, dollars*
and x = *number of miles driven*
This equation is in slope-intercept form with a slope of 0.40 and C intercept (which is equivalent to the y intercept) of (0, 18,000). The slope suggests that total cost increases at a rate of $0.40 for each additional mile driven. The C intercept indicates a cost of $18,000 if the car is driven zero miles.

□

Section 2.3 Follow-up Exercises

For Exercises 1–24, rewrite each equation in slope-intercept form and determine the slope and y intercept.

1 $3x - 2y = 15$
2 $-x + 5y = 27.5$
3 $4x - 3y = 18$
4 $2x - 7y = -21$
5 $-x + y = 8$
6 $2x - y = -5$
7 $(x + 2y)/2 = -6$
8 $(-2x + y)/3 = 2$
9 $(3x - 5y)/4 = -5$
10 $(-x + 2y)/4 = 3x - y$

11 $2x = (5x - 2y)/4$
12 $(-x + 3y)/2 = 10 - 2x$
13 $4x - 3y = 0$
14 $8x + 3y = 24$
15 $3x - 6y + 10 = x$
16 $3y - 5x + 20 = 4x - 2y + 5$
17 $2x + 3y = 4x + 3y$
18 $-5x + y - 12 = 2y - 5x$
19 $8y - 24 = 0$
20 $3x + 6 = 0$
21 $mx + ny = p$
22 $mx - n = 0$
23 $c - dy = 0$
24 $dx = cy - f$

25 **Women in the Labor Force** The number of women in the labor force is expected to increase during the 1990s, but not as dramatically as occurred during the 1970s. One forecasting consultant uses the linear equation $n = 29.6 + 1.20t$ to predict the number of women between the ages of 35 and 44 who will be in the labor force. In this equation, n equals the number of women (aged 35 to 44) in the labor force (measured in millions) and t equals time measured in years *since* 1981 ($t = 0$ corresponds to 1981). If n is plotted on the vertical axis:
(*a*) Graph the equation.
(*b*) Identify the slope and y intercept (n intercept, here).
(*c*) Interpret the meaning of the slope and n intercept in this application.
(*d*) Predict the number of women in this age group who will be in the labor force in 1995. In the year 2000.

26 The chamber of commerce for a summer resort is trying to determine how many tourists will be visiting each season over the coming years. A marketing research firm has estimated that the number of tourists can be predicted by the equation $p = 275{,}000 + 7{,}500t$, where p = number of tourists per year and t = years (measured *from* this current season). Thus $t = 0$ identifies the current season, $t = 1$ is the next season, etc. If p is plotted on the vertical axis:
(*a*) Graph the equation.
(*b*) Identify the slope and y intercept (p intercept, here).
(*c*) Interpret the meaning of the slope and p intercept in this application.

27 **Think Metric!** $C = \frac{5}{9}F - \frac{160}{9}$ is an equation relating temperature in Celsius units to temperature measured on the Fahrenheit scale. Let C = degrees Celsius and F = degrees Fahrenheit; assume the equation is graphed with C measured on the vertical axis.
(*a*) Identify the slope and C intercept.
(*b*) Interpret the meaning of the slope and C intercept for purposes of converting from Fahrenheit to Celsius temperatures.
(*c*) Solve the equation for F and rework parts *a* and *b* if F is plotted on the vertical axis.

28 A police department believes that the number of serious crimes which occur each month can be estimated by the equation

$$c = 1{,}200 - 12.5p$$

where c equals the number of serious crimes expected per month and p equals the number of officers assigned to preventive patrol. If c is graphed on the vertical axis:
(*a*) Identify the slope and interpret its meaning.
(*b*) Identify the c intercept and interpret its meaning.
(*c*) Identify the p intercept and interpret its meaning.

29 The book value of a machine is expressed by the equation

$$V = 60{,}000 - 7{,}500t$$

where V equals the book value in dollars and t equals the age of the machine expressed in years.

(a) Identify the t and V intercepts.
(b) Interpret the meaning of the intercepts.
(c) Interpret the meaning of the slope.
(d) Sketch the function.

30 **SAT Scores** One small college has observed a downward trend in the average SAT score of applicants to the college. Analysis has resulted in the equation

$$s = 620 - 4.5t$$

where s equals the average SAT score for a given year and t equals time measured in years since 1985 ($t = 0$).
(a) Identify the t and s intercepts.
(b) Interpret the meaning of the intercepts. (Does your interpretation of the t intercept make sense?)
(c) Interpret the meaning of the slope.
(d) Sketch the equation.

31 **Product Mix** A company produces two products. Weekly labor availability equals 150 labor-hours. Each unit of product 1 requires 3 labor-hours and each unit of product 2 requires 4.5 labor-hours. If management wishes to use all labor-hours, the equation

$$3x + 4.5y = 150$$

is a statement of this requirement, where x equals the number of units produced of product 1 and y equals the number of units produced of product 2. Rewrite the equation in slope-intercept form and interpret the meaning of the slope and y intercept. Solve for the x intercept and interpret its meaning.

32 **Portfolio Management** A portfolio manager is concerned that two stocks generate an annual income of \$15,000 for a client. The two stocks earn annual dividends of \$2.40 and \$3.50 per share, respectively. If x equals the number of shares of stock 1 and y equals the number of shares of stock 2, the equation

$$2.4x + 3.5y = 15{,}000$$

states that total annual dividend income from the two stocks should total \$15,000. Rewrite the equation in slope-intercept form and interpret the meaning of the slope and y intercept in this application. Solve for the x intercept and interpret its meaning.

2.4 DETERMINING THE EQUATION OF A STRAIGHT LINE

In this section we show how to determine the equation for a linear relationship. The way in which you determine the equation depends upon the information available. The following sections discuss different possibilities. In each instance we will be seeking the slope-intercept form. Thus, we will need to identify the slope-intercept *parameters* m and k.

Slope and Intercept

The easiest situation is one in which you know the slope m and y intercept $(0, k)$ of the line representing an equation. To determine the linear equation in this almost trivial case, simply substitute m and k into the slope-intercept form, Eq. (2.12). I

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you are interested in stating the equation in the standard form of Eq. (2.1), simply rearrange the terms of the slope-intercept equation.

EXAMPLE 13 Determine the equation of the straight line which has a slope of −5 and a y intercept of (0, 15).

SOLUTION

Substituting values of $m = -5$ and $k = 15$ into Eq. (2.12) gives

$$y = -5x + 15$$

Restated in the form of Eq. (2.1), an equivalent form of this equation is

$$5x + y = 15$$

❑

Slope and One Point

If given the slope and one point which lies on a line, we can substitute the known slope m and coordinates of the given point into Eq. (2.12) and solve for k.

EXAMPLE 14 Given that the slope of a straight line is −2 and one point lying on the line is (2, 8) we can substitute these values into Eq. (2.12), yielding

$$8 = (-2)(2) + k$$

or

$$12 = k$$

Since $m = -2$ and $k = 12$, the slope-intercept equation is

$$y = -2x + 12$$

And, as before, we can rewrite this equation in the equivalent form

$$2x + y = 12$$

❑

NOTE You may be wondering which form of the linear equation — Eq. (2.1) or Eq. (2.12) — is the correct or preferred form. Both are correct! The preferred form depends on what you intend to do with the equation. Depending on the type of analysis to be conducted, one of these forms may be more appropriate than the other.

$$-30 = (0)(5) + k$$

or

$$-30 = k$$

Since $m = 0$ and $k = -30$, the slope-intercept equation is

$$y = 0x + (-30)$$

or

$$y = -30$$

EXAMPLE 16 **(Point-Slope Formula)** Given a nonvertical straight line with slope m and containing the point (x_1, y_1), the slope of the line connecting (x_1, y_1) with any other point (x, y) on the line would be expressed as

$$m = \frac{y - y_1}{x - x_1}$$

Rearranging this equation results in

$$y - y_1 = m(x - x_1) \tag{2.13}$$

which is the ***point-slope formula*** for a straight line. This formula can be used to determine the equation of a nonvertical straight line given the slope and one point lying on the line. Suppose that a line has a slope of 5 and contains the point $(-4, 10)$. Substituting into Eq. (2.13) and solving for y,

$$y - 10 = 5[x - (-4)]$$

$$y - 10 = 5x + 20$$

$$y = 5x + 30$$

which is the slope-intercept form of the equation.

EXAMPLE 17 Considering the linear equation $3x - 6y = 24$:

(*a*) What is the slope of the line represented by the given equation?
(*b*) What is the slope of any line parallel to the given line?
(*c*) What is the slope of any line perpendicular to the given line?
(*d*) How many different lines are perpendicular to this line?
(*e*) Find the equation of the line which is perpendicular to the given line *and* which passes through the point (2, 5).

SOLUTION

(*a*) The given equation can be restated in slope-intercept form as

$$-6y = 24 - 3x$$

or

$$y = -4 + \tfrac{1}{2}x$$

From this equation the slope equals $+\frac{1}{2}$, and the y intercept occurs at $(0, -4)$.

(*b*)

PARALLEL LINES
Two lines are parallel if they have the same slope.

Since the slope of the given line equals $+\frac{1}{2}$, any parallel lines will have a slope of $+\frac{1}{2}$.

(*c*)

PERPENDICULAR LINES
If a line has a slope m_1 ($m_1 \neq 0$), the slope of any line perpendicular to the given line has a slope equal to the negative reciprocal of the given line, or $m_2 = -1/m_1$.

Since $m_1 = \frac{1}{2}$, the slope of any line perpendicular to the line $3x - 6y = 24$ is

$$m_2 = -\frac{1}{\frac{1}{2}}$$
$$= -2$$

(*d*) Because there is an infinite set of lines with $m = -2$, an infinite number of lines are perpendicular to this line.
(*e*) The line we are interested in has a slope equal to -2 and one point on the line is (2, 5). Substituting these values into the point-slope formula, Eq. (2.13), gives

$$y - 5 = (-2)(x - 2)$$
$$y - 5 = -2x + 4$$
$$y = -2x + 9$$

or, alternatively,

$$2x + y = 9$$

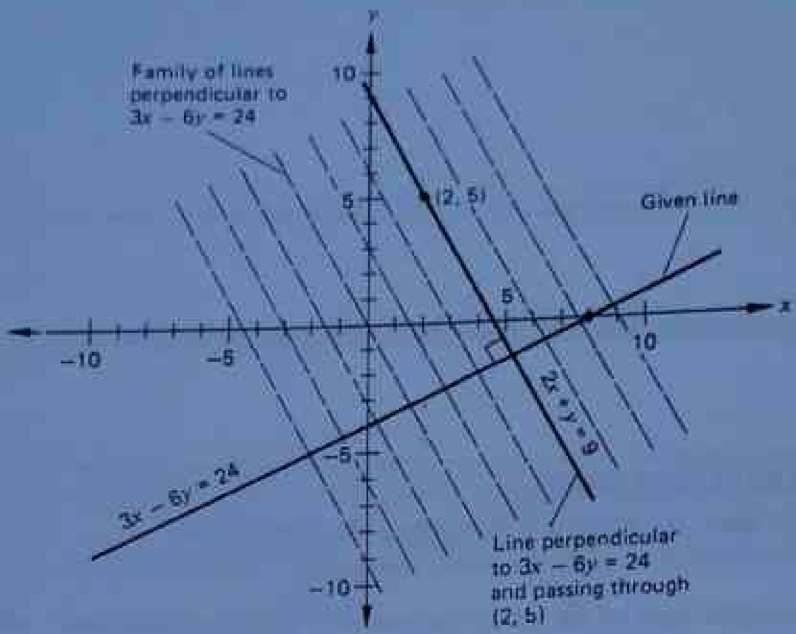
Figure 2.11 illustrates the lines in this example.

❑

Two Points

A more likely situation is that some data points which lie on a line have been gathered and we wish to determine the equation of the line. Assume that we are given the coordinates of two points which lie on a straight line. We can determine the slope of the line by using the two-point formula [Eq. (2.10)]. As soon as we know the slope, the y intercept can be determined by using *either* of the two data points and proceeding as we did in the previous section.

Figure 2.11



EXAMPLE 18 To determine the equation of the straight line which passes through (−4, 2) and the origin, we substitute the coordinates into the two-point formula, resulting in

$$m = \frac{0-2}{0-(-4)}$$

$$= \frac{-2}{4} = -\frac{1}{2}$$

Substituting $m = -\frac{1}{2}$ and the coordinates (−4, 2) into Eq. (2.13) yields

$$y - 2 = (-\tfrac{1}{2})[x - (-4)]$$

$$y - 2 = -\tfrac{1}{2}x - 2$$

$$y = -\tfrac{1}{2}x$$

Thus, the slope-intercept form of the equation is

$$y = -\tfrac{1}{2}x$$

□

NOTE In this last example you might have realized that the origin is the y intercept. How would this have simplified the analysis?

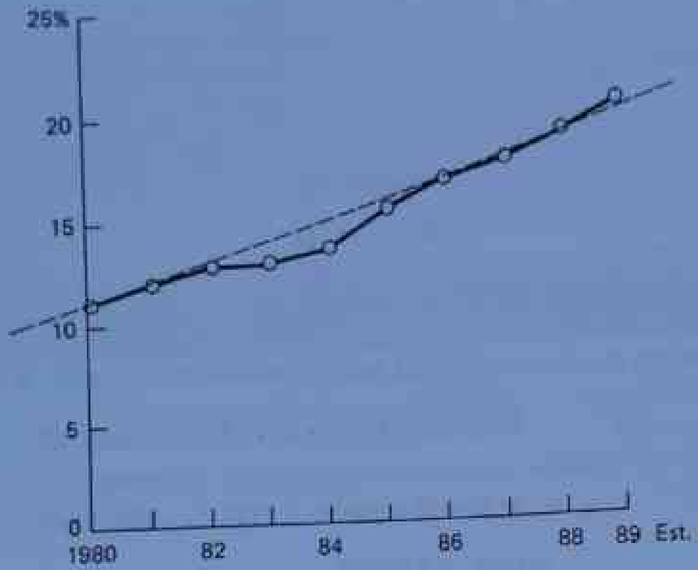


Figure 2.12 Percentage of total electricity produced in the United States attributable to nuclear sources. (Sources: Chicago Tribune, North American Electricity Council.)

EXAMPLE 19 **(Nuclear Power Utilization; Motivating Scenario)** Figure 2.12 illustrates sample data gathered by the North American Electricity Council. The graph illustrates the percentage of total electricity produced in the United States generated by nuclear power sources. The percentage appears to be increasing approximately at a linear rate over time. The council wants to determine a linear equation which approximates the relationship between the percentage of electricity generated by nuclear sources (p) and time (t) measured in years. An analyst has chosen to fit a line through the data points for 1981 and 1986. The values for p were 0.124 and 0.172 for the years 1981 and 1986, respectively.

In determining the estimating equation, let t equal years measured since 1980 (i.e., $t = 0$ corresponds to 1980, $t = 1$ corresponds to 1981, etc.). Using this definition, the two data points have coordinates (1, 0.124) and (6, 0.172). The slope-intercept relationship for this example will have the form

$$p = mt + k \tag{2.14}$$

Given the two data points, the slope is

$$\begin{aligned} m &= \frac{p_2 - p_1}{t_2 - t_1} \\ &= \frac{0.172 - 0.124}{6 - 1} \\ &= \frac{0.048}{5} = 0.0096 \end{aligned}$$

Using a slightly different, but equivalent, procedure to the point-slope formula, we substitute $m = 0.0096$ and the coordinates (1, 0.124) into Eq. (2.14):

$$0.124 = (0.0096)(1) + k$$

$$0.1144 = k$$

Therefore, the slope-intercept form of the estimating equation is

$$p = 0.0096t + 0.1144$$

EXAMPLE 20 **(Nuclear Power Utilization, continued)** In the last example:
(*a*) Interpret the meaning of the slope and p intercept.
(*b*) According to this estimating equation, what percentage of electricity is expected from nuclear sources in the year 2000?
(*c*) According to this equation, when will the percentage surpass 25 percent?

SOLUTION

(*a*) The slope indicates that for each additional year, the percentage of electricity attributable to nuclear sources increases by 0.0096, or by 0.96 percent. The p intercept indicates that the estimated percentage for the year 1980 was 0.1144, or 11.44 percent.
(*b*) A t value of 20 corresponds to the year 2000. Substituting this into the estimating equation gives

$$\begin{aligned} p &= 0.0096(20) + 0.1144 \\ &= 0.192 + 0.1144 \\ &= 0.3064 \end{aligned}$$

This estimating equation predicts that 30.64 percent will be attributable to nuclear sources in the year 2000.
(*c*) Letting $p = 0.25$ yields

$$\begin{aligned} 0.25 &= 0.0096t + 0.1144 \\ 0.1356 &= 0.0096t \\ \frac{0.1356}{0.0096} &= t \\ 14.125 &= t \end{aligned}$$

Therefore, when $t = 14.125$, the percentage will equal 0.25. Therefore, the percentage will surpass 25 percent sometime during the fifteenth year, or during 1995.

NOTE Estimation plays a very important part in applying mathematical analysis to the world around us. Although the tools of mathematical analysis are most often very precise, the relationships which we analyze are not always exact. There are many applications in which the mathematical relationships are determined precisely. However, we often must estimate the relationships which exist between variables that are of interest. There are scientific procedures which can be used to develop our estimates. The use of such procedures enhances the likelihood that our estimates are reasonable. However, as a person actually conducting mathematical analysis or as a person who is the recipient of the results of such analyses, one should be aware that estimated relationships are usually accompanied by some measure of error. An attempt should be made to understand the magnitude of potential error which is associated with estimates and to consider the effects of such error in drawing conclusions from the mathematical analysis.

As we move through the text, you will sometimes develop mathematical relationships and sometimes be given relationships. Try to retain a questioning attitude about the source of each relationship. Be curious about their origins. Anticipate the implications of errors in the relationships. The author will try to reinforce this perspective.

Section 2.4 Follow-up Exercises

In Exercises 1–36, determine the slope-intercept form of the linear equation, given the listed attributes.

1 Slope = −2, y intercept = (0, 10)
2 Slope = 4, y intercept = (0, −5)
3 Slope = $\frac{1}{2}$, y intercept = (0, $\frac{3}{4}$)
4 Slope = $-\frac{8}{3}$, y intercept = (0, −20)
5 Slope = $-r$, y intercept = (0, $-t/2$)
6 Slope undefined, infinite number of y intercepts
7 Slope = −3, (4, −2) lies on line
8 Slope = 5, (−3, 12) lies on line
9 Slope = $\frac{3}{2}$, (−5, −8) lies on line
10 Slope = $-\frac{1}{2}$, (−4, 0) lies on line
11 Slope = 2.5, (−2, 5) lies on line
12 Slope = −3.25, (1.5, −7.5) lies on line
13 Slope = 5.6, (2.4, −4.8) lies on line
14 Slope = −8.2, (−0.75, 16.3) lies on line
15 Slope = w, (p, q) lies on line
16 Slope = $-a$, (4, −4) lies on line
17 Slope undefined, (−3, −5) lies on line
18 Slope = 0, (20, −10) lies on line
19 Slope = 0, (u, v) lies on line
20 Slope undefined, ($-t$, v) lies on line
21 (−4, 5) and (−2, −3) lie on line
22 (3, −2) and (−12, 1) lie on line
23 (20, 240) and (15, 450) lie on line
24 (−12, 760) and (8, −1,320) lie on line
25 (0.234, 20.75) and (2.642, 18.24) lie on line
26 (5.76, −2.48) and (3.74, 8.76) lie on line
27 (a, b) and (c, d) lie on line
28 (a, −3) and (a, 15) lie on line
29 ($-d$, b) and (e, b) lie on line
30 (p, r) and ($-p$, r) lie on line
31 Passes through (2, −4) and is parallel to the line $3x - 4y = 20$
32 Passes through (−2, 10) and is parallel to the line $5x - y = 0$
33 Passes through (7, 2) and is parallel to the line (a) $x = 7$ and (b) $y = 6$
34 Passes through (20, −30) and is perpendicular to the line $4x + 2y = -18$
35 Passes through (−8, −4) and is perpendicular to the line $8x - 2y = 0$
36 Passes through (7, 2) and is perpendicular to the line (a) $x = 7$ and (b) $y = 6$

37 **Depreciation** The value of a machine is expected to decrease at a linear rate over time. Two data points indicate that the value of the machine at $t = 0$ (time of purchase) is \$18,000 and its value in 1 year will equal \$14,500.

(*a*) Determine the slope-intercept equation ($V = mt + k$) which relates the value V of the machine to its age t.

(*b*) Interpret the meaning of the slope and V intercept.

(*c*) Solve for the t intercept and interpret its meaning.

38 **Depreciation** The value of a machine is expected to decrease at a linear rate over time. Two data points indicate that the value of the machine 1 year after the date of purchase will be \$84,000 and its value after 5 years is expected to be \$36,000.

(*a*) Determine the slope-intercept equation ($V = mt + k$) which relates the value V of the machine to its age t, stated in years.

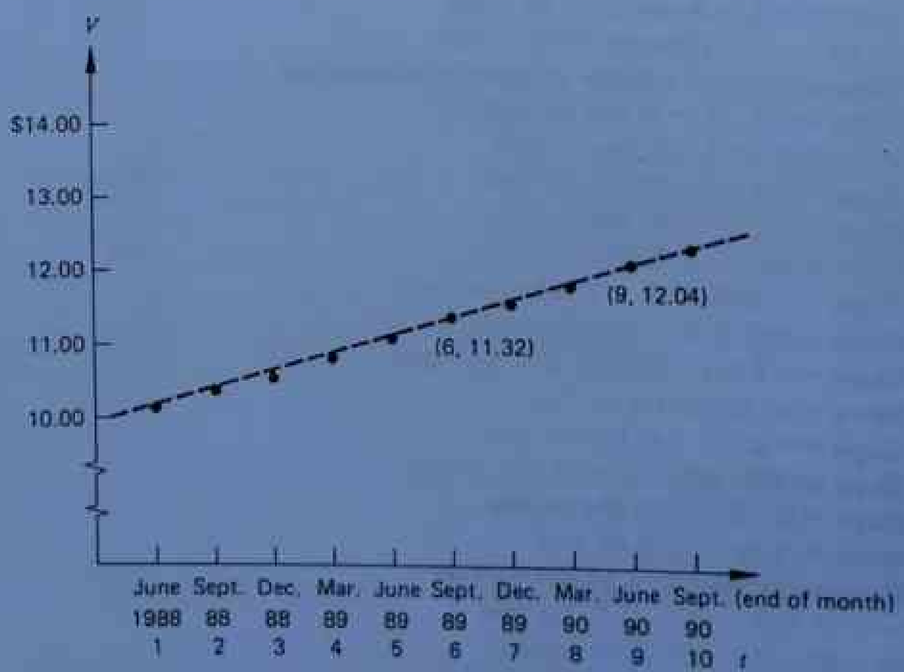
(*b*) Interpret the meaning of the slope and V intercept.

(*c*) Determine the t intercept and interpret its meaning.

39 If C equals degrees Celsius and F equals degrees Fahrenheit, assume that the relationship between the two temperature scales is linear and is being graphed with F on the vertical axis. Two data points on the line relating C and F are (5, 41) and (25, 77). Using these points, determine the slope-intercept equation which allows transformation from Celsius temperature to equivalent Fahrenheit temperature. Identify and interpret the meaning of the slope, C intercept, and F intercept.

40 **College Retirement** The largest retirement program for college professors is the Teachers Insurance and Annuity Association/College Retirement Equities Fund (TIAA/CREF). One of the investment options in this program is the CREF Money Market Account, which was initiated in 1988. Figure 2.13 illustrates the performance of this investment during the first 10 quarters of its existence. Note that V is the value of a share (unit) in this fund and that the data points reflect the end-of-month values. It appears that the value of this money market has been increasing at an approximately linear rate. If the data points (6, 11.32) and (9, 12.04) are chosen to estimate the relationship between the value of a share V and time t, measured in quarters since the inception of the CREF Money Market Fund ($t = 0$ corresponds to March 31, 1988):

Figure 2.13 CREF money market account per share (unit) quarter-end values.



(a) Determine the slope-intercept form of the estimating equation.
(b) Identify and interpret the meaning of the slope.
(c) Forecast the value per share on June 30, 1991, and March 31, 1992.

41 **College Retirement (continued)** The CREF Money Market Fund was established on March 31, 1988. The initial value per share was set at $10.00.
(a) Using the equation found in part *a* of the previous exercise, estimate the value per share on March 31, 1988. How much error is there in the estimate?
(b) Similarly, determine the actual values per share on June 30, 1991, and March 31, 1992,* and compare with the forecasts in part *c* of the previous exercise. How much error was there?
(c) While you have access to the data in part *b*, test the accuracy of the estimating equation for other quarterly data points.

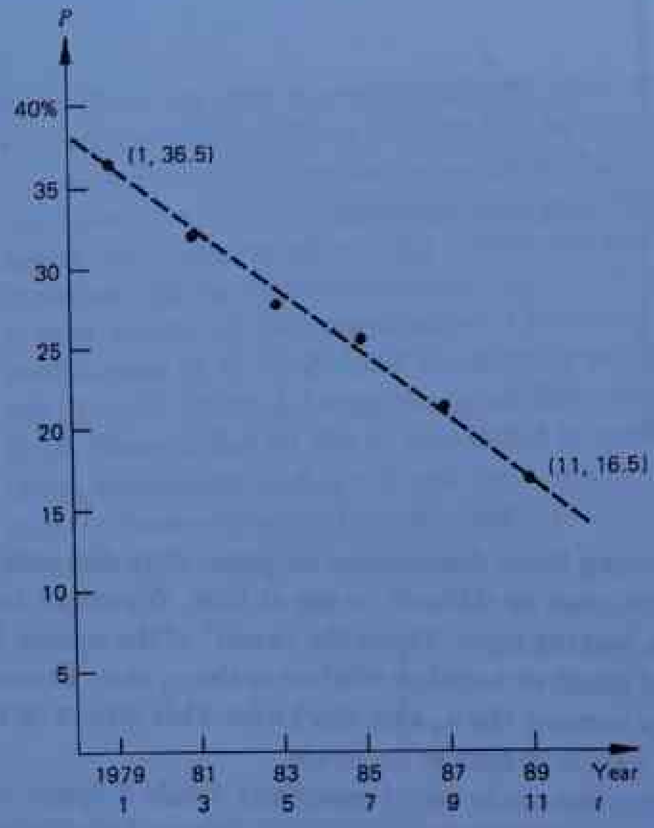


Figure 2.14 Percentage of high school students who have used marijuana in the past 30 days.

42 **Marijuana Usage among High School Students** Figure 2.14 illustrates some survey data regarding the usage of marijuana among high school students. A sample of high school students was taken every 2 years between 1979 and 1989. The data in Figure 2.14 reflects the percentage of students surveyed who indicated they had used marijuana during the previous 30 days. The data points suggest that the percentage of students having used marijuana is decreasing at an approximately linear rate over time. If the

* Contact the Teachers Insurance and Annuity Association/College Retirement Equities Fund, 730 Third Avenue, New York, New York 10017 (try calling 1-800-842-2733).

data points for 1979 (1, 36.5) and 1989 (11, 16.5) are used to estimate the linear equation which relates the percentage of students P to time t ($t = 1$ corresponding to 1979):

(a) Determine the slope-intercept form of the estimating equation.

(b) Forecast the expected percentage for 1991 and 1995.

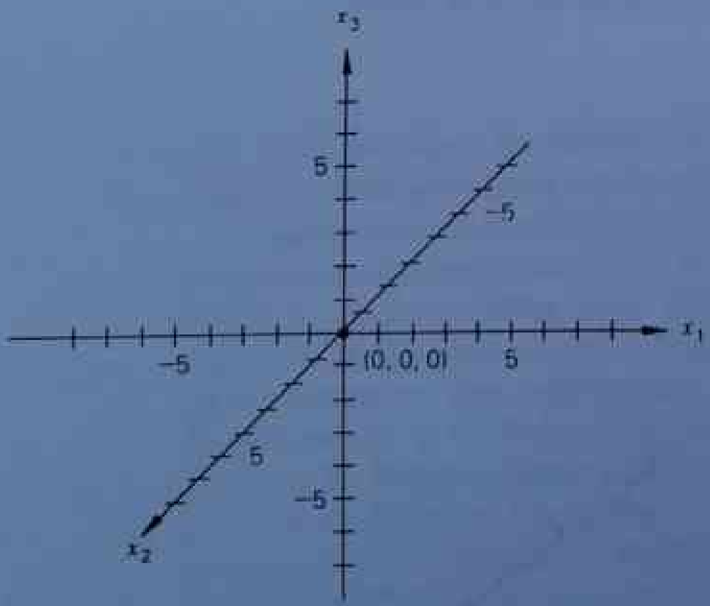
(c) Interpret the meaning of the slope and P intercept.

2.5 LINEAR EQUATIONS INVOLVING MORE THAN TWO VARIABLES

When linear equations involve more than two variables, the algebraic properties remain basically the same but the visual or graphical characteristics change considerably or are lost altogether.

Three-Dimensional Coordinate Systems

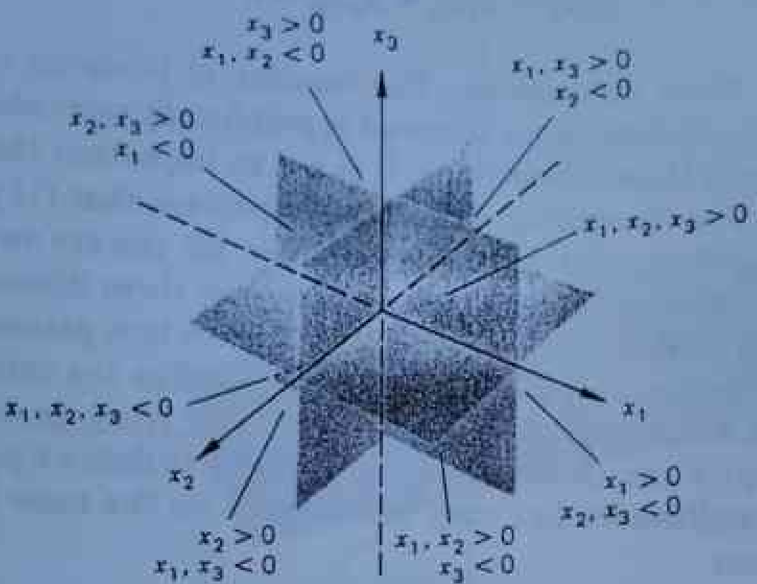
Three-dimensional space can be described by using a ***three-dimensional coordinate system.*** In three dimensions we use three coordinate axes which are all perpendicular to one another, intersecting at their respective zero points. Figure 2.15 illustrates a set of axes which are labeled by the variables x_1, x_2, and x_3. The point of intersection of the three axes is referred to as the ***origin.*** Using three-component coordinates ***(ordered triples),*** (x_1, x_2, x_3), the coordinates of the origin are (0, 0, 0).



Observe that graphing three dimensions on paper (two dimensions) requires a certain perspective that may be difficult to see at first. We might have drawn Fig. 2.15 such that we were looking right "down the barrel" of the x_2 axis. In that case we would have no sense of depth or location relative to the x_2 axis. Therefore, we rotate the coordinate axes by turning the x_3 axis clockwise.This allows us to have a sense of depth when the x_2 axis is drawn at an angle.

Just as the coordinate axes in two dimensions divide 2-space into quadrants, the axes in three dimensions divide 3-space into ***octants.*** This is illustrated in Fig.

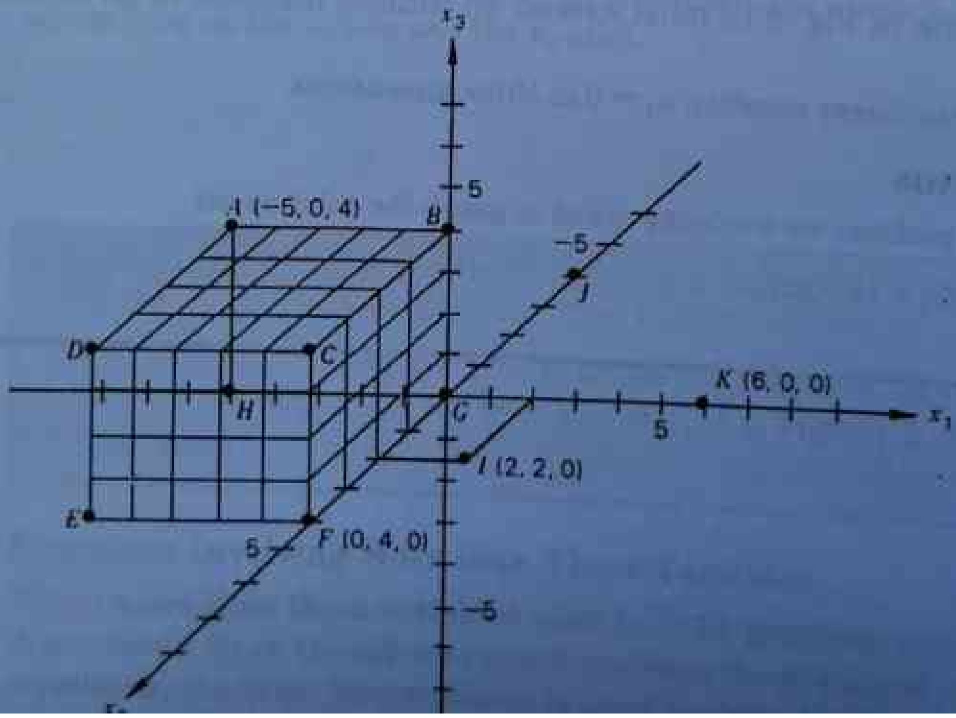
16
f



2.16. Note the sign characteristics in each octant. The three-component coordinates allow for specifying the location or address of any point in three dimensions.

As with two-dimensional coordinates, each component of (x_1, x_2, x_3) specifies the location of a point relative to each axis. Carefully examine Fig. 2.17. In order to assist in understanding this figure, a ***rectangular polyhedron*** has been sketched. Along with several other points, we are interested in the locations of the corner points of this polyhedron. Obviously, G is located at the origin, having coordinates (0, 0, 0). Point F lies directly on the x_2 axis, 4 units out. Its coordinates are (0, 4, 0). Point A forms the upper left corner of one end ($ABGH$) of the polyhedron. Since H lies on the x_1 axis and A is vertically above H, we can conclude that the x_1 coordinate of A is -5 *and* the x_2 coordinate of A is 0. Finally, points A, B, C, and D all seem to be at the same *height* (relative to the x_3 axis). Because point B lies

17
oints
e.



on the x_3 axis at a height of 4, we conclude that A has the same x_3 coordinate. Thus, A is located at $(-5, 0, 4)$. See if you agree with the coordinates of I and k.

PRACTICE EXERCISE

Test your skills and define the coordinates of points B, C, D, E, and J. *Answer:* $B(0, 0, 4)$, $C(0, 4, 4)$, $D(-5, 4, 4)$, $E(-5, 4, 0)$, $J(0, -4, 0)$.

Equations Involving Three Variables

Linear equations having the form

$$a_1x_1 + a_2x_2 + a_3x_3 = b$$

graph as ***planes*** in three dimensions. *The number of variables in an equation determines the number of dimensions required to graphically represent the equation.* Three variables require three dimensions. It is not so important that you actually be able to graph in three dimensions. It is more important that (1) you are able to *recognize* a linear equation involving three variables, (2) you are aware that linear equations involving three variables graph as planes in three dimensions, (3) you know what a *plane* is, and (4) you have some feeling for how planes can be represented graphically. A plane, of course, is a flat surface like the ceiling, walls, and floor of the room in which you are currently sitting or lying. Instead of the two points needed to graph a line, three points are necessary to define a plane. The three points must not be ***collinear;*** i.e., they must not lie on the same line. Take, for example, the equation

$$2x_1 + 4x_2 + 3x_3 = 12 \qquad \textbf{(2.15)}$$

If we can identify three members of the solution set for this equation, they will specify the coordinates of three points lying on the plane. Three members which are identified easily are the intercepts. These are found by setting any two of the three variables equal to 0 and solving for the remaining variable. Verify that when $x_1 = x_2 = 0$, $x_3 = 4$, or $(0, 0, 4)$ is a member of the solution set. Similarly, verify that $(6, 0, 0)$ and $(0, 3, 0)$ are members of the solution set and thus are points lying on the plane representing Eq. (2.15). Figure 2.18 shows these points and a portion of the plane which contains them.

When graphing equations involving two variables, we identified two points and connected them with a straight line. However, we saw that in order to represent *all* members of the solution set, the line must extend an infinite distance in each direction. The same is true with the solution set for three-variable equations. To represent all members of the solution set for the equation $2x_1 + 4x_2 + 3x_3 = 12$, the plane in Fig. 2.18 must extend an infinite distance in all directions.

21 Graph the linear equation $x_1 = 0$ in three dimensions.

SOLUTION

In this problem we are being asked to graph the solution set

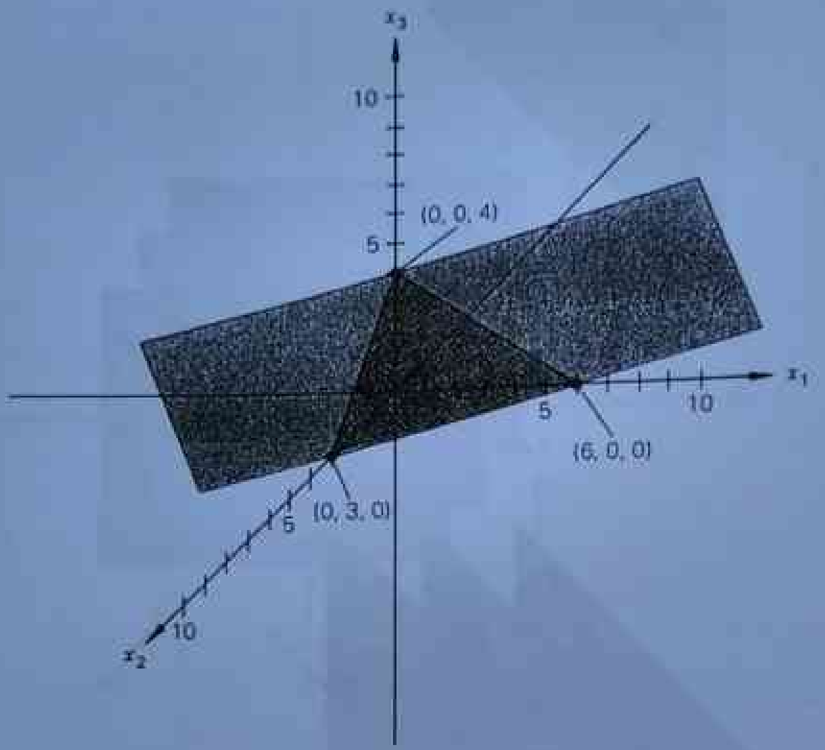


Figure 2.18 Graph of plane representing the linear equation $2x_1 + 4x_2 + 3x_3 = 12$.

$$S = \{(x_1, x_2, x_3) | x_1 = 0\}$$

In order to graph the equation, we again need to identify three noncollinear points which satisfy the equation. We see that as long as $x_1 = 0$, x_2 and x_3 can equal *any* values. For example, (0, 0, 0), (0, 2, 0), and (0, 0, 4) all satisfy the equation. Figure 2.19 illustrates the graph of the equation. The equation $x_1 = 0$ graphs as a plane perpendicular to the x_1 axis and passing through $x_1 = 0$. This is the x_2x_3 *plane* (the plane which includes among its points all points lying on the x_2 axis *and* the x_3 axis).

❑

> Any equation of the form $x_1 = k$ graphs in 3-space as a plane perpendicular to the x_1 axis, intersecting it at $x_1 = k$.
>
> Any equation of the form $x_j = k$, where $j = 1, 2,$ *or* 3, will graph as a plane which is perpendicular to the x_j axis at $x_j = k$. Figures 2.20 to 2.22 illustrate this property.

Equations Involving More than Three Variables

When more than three variables exist ($n > 3$), graphing requires more than three dimensions. Even though we cannot envision the graphical representation of such equations, the term ***hyperplane*** is used to describe the ("would-be") geometric

Figure 2.19
The plane $x_1 = 0$.

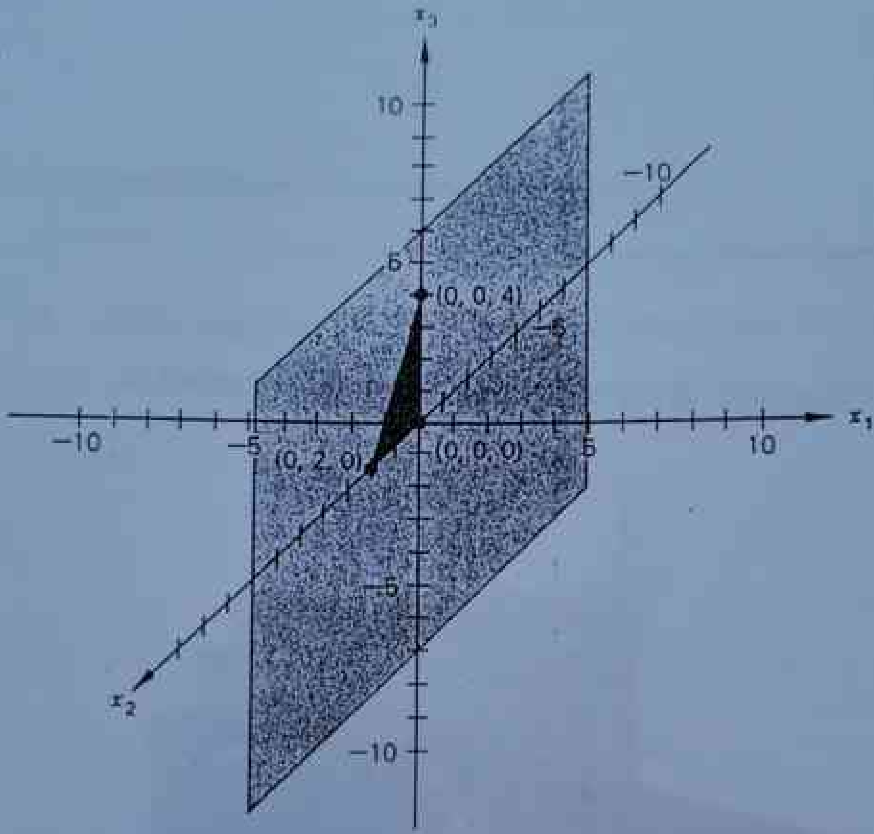


Figure 2.20
Planes of the form $x_1 = k$.

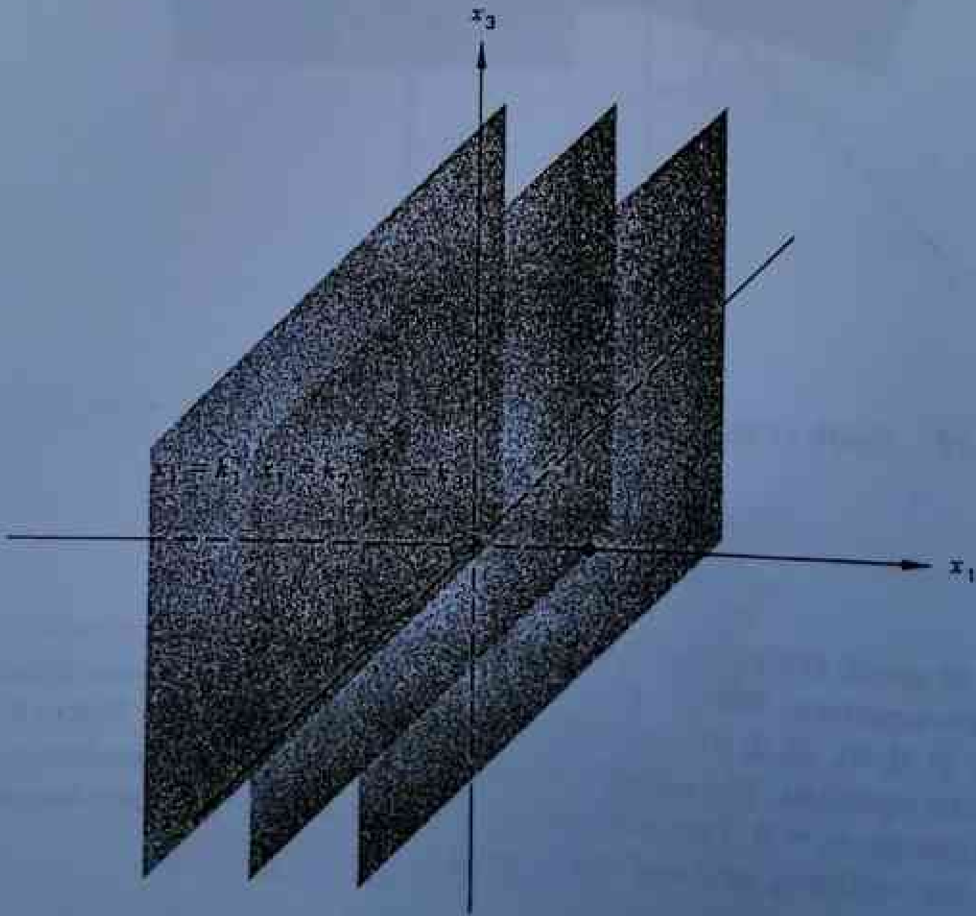


Figure 2.21
Planes of the form $x_2 = k$.

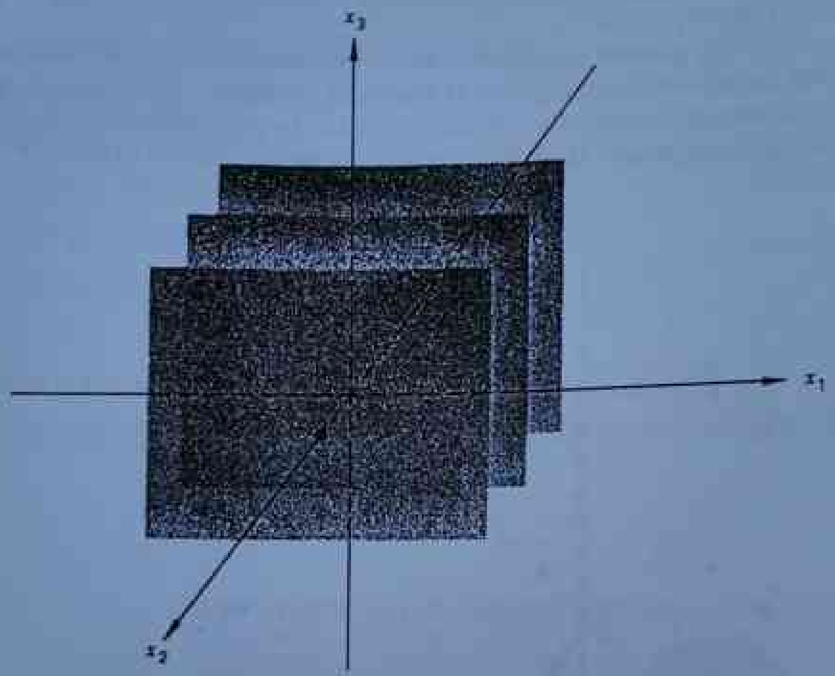
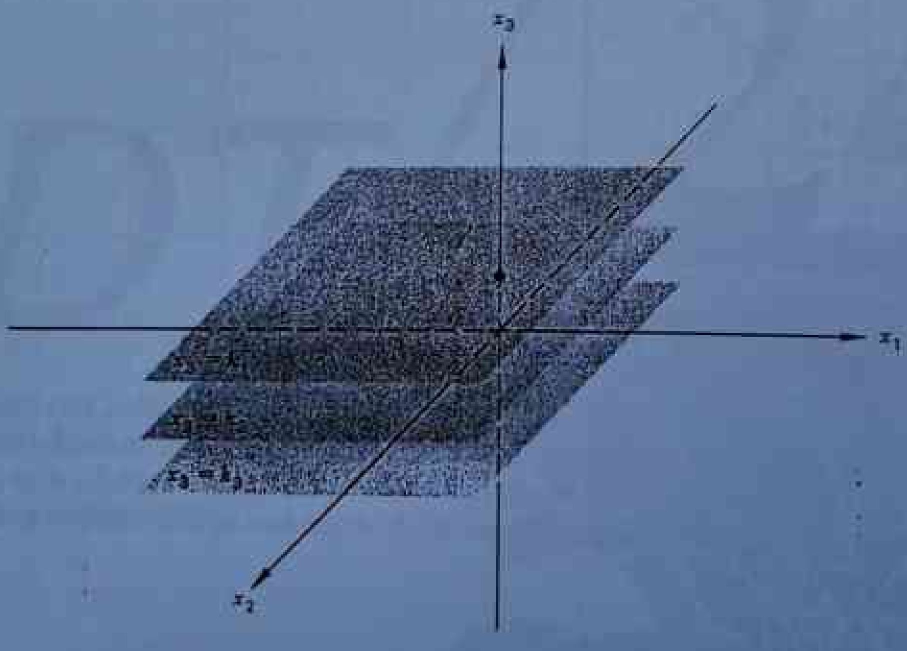


Figure 2.22
Planes of the form $x_3 = k$.



representation of the equation. Mathematicians would, for instance, say that the equation

$$x_1 + x_2 + x_3 + x_4 = 10$$

is represented by a hyperplane in 4-space or four dimensions. Or, in general, an equation of the form

EXAM

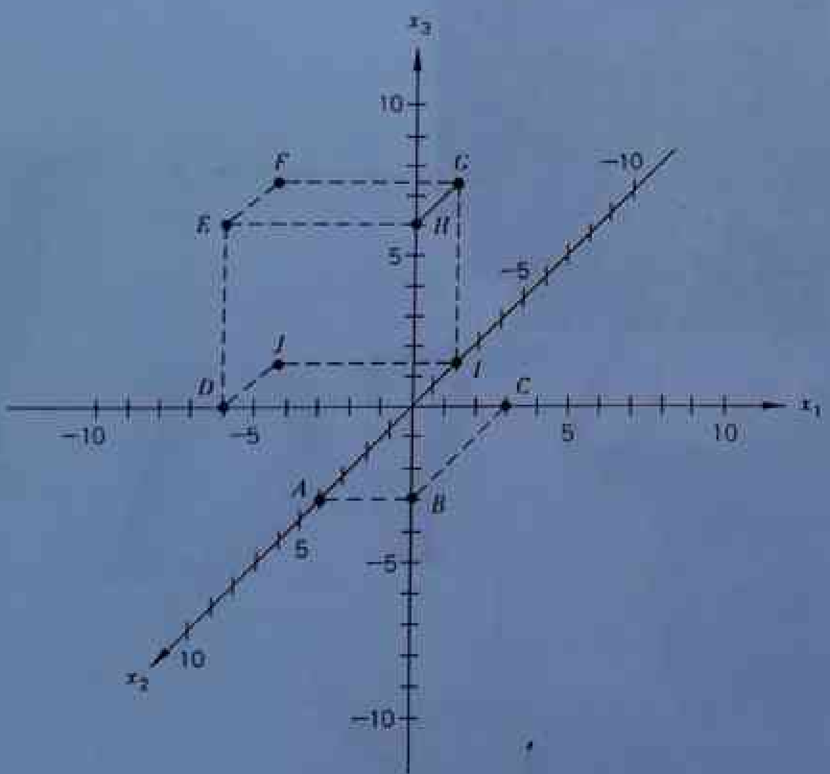
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

where $n > 3$, would be represented by a hyperplane in ***n*-space.**

Section 2.5 Follow-up Exercises

1 Given Fig. 2.23, determine the coordinates of points *A* through *I*.
2 Given the equation $x_1 - 2x_2 + 4x_3 = 10$, determine the coordinates of the x_1, x_2, and x_3 intercepts.
3 Given the equation $-2x_1 + 3x_2 - x_3 = -15$, determine the coordinates of the x_1, x_2, and x_3 intercepts.
4 Sketch the plane $3x_1 = 9$.
5 Sketch the plane $-2x_2 = -8$.
6 Sketch the plane $x_3 = -2$.
•7 Can you draw any general conclusions about the characteristics of planes which represent linear equations involving two of the three variables? For example, the equation $x_1 + x_2 = 5$ does not contain the variable x_3 but can be graphed in three dimensions. How does this equation graph? How about equations which involve x_1 and x_3? x_2 and x_3?

2.23



2.6 ADDITIONAL APPLICATIONS

The more exposure you have to word problems, the more skilled you will become in formulating them. The following examples illustrate the formulation of linear equations for different types of applications. Study these carefully and try as many of these types of problems as you can, both at the end of this section and at the end of the chapter.